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EASILY COMPREHENSIBLE MATHEMATICAL LOGIC AND ITS MODEL THEORY

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KIVONAT

A tanulmány ujszerü közelitésmódban tárgyalja a matematikai logikát. Részletesen foglalkozik a nullad-, első- és másodrendü nyelvekkel és logikákkal. E logikák bizonyitáselméletének és modellelméletének is részletes leirását adja. Több alapvető tételnek pl. a teljességi tételnek uj, egyszerübb bizonyitása található benne. A tanulmány különösen hasznos anyagot nyujt a számitástudomány azon szakemberei számára, akik magasszintü programozási nyelvek, valamint automatikus tételbizonyitó programrendszerek kifejlesztésén tevékenykednek.

ABSTRACT

Present study discusses mathematical logic in a new approach. The zero-, first- and second-order languages and logics are discussed in details. Their proof theory and model theory are also described. New and easily comprehensible proofs are given for the basic theorem e.g. completeness theorem. It provides useful material to those engaged in the field of computing sciences especially in the development of high-level programming languages and automatic theorem proving program systems.

АННОТАЦИЯ

Настоящая работа посвящена изложению математической логики с новой точки зрения. Подробно излагаются языки и логики 0-го, I-го и 2-го порядка, вместе с их теорией доказательств и теорией моделей. Приводятся новые и легко доступные доказательства основных теорем, так например теоремы полноты. Работа представляет интерес для тех специалистов науки вычислений, которые работают в области создания языков программирования высокого уровня и программ для автоматического доказательства.

I. INTRODUCTION

In the present paper dealing with purely mathematical investigation of both classical first-order language and logic, the intuitive mathematical language will also be used, for clearer comprehension, as meta-language.
In other words, the intuitive interpretation of concrete mathematical objects appearing in the mathematical construction of the logic under investigation is described using the English terminology of mathematics.
As it is well known, language and logic, are closely connected with human thinking, more precisely with the activities of problem solving, knowledge aquisition, etc.
Mathematical logic on the other hand is the mathematical study of these activities as they are connected to language and logic.

Let us first consider the intuitive, everyday interpretation of language. Language is the means of cognition of objective reality. In that the world manifestly is diversified and complex, thus a set of possible worlds becomes existent relative to the subjective mind during the course of cognition.
Let us now investigate the role of language in the cognitive process. Language must be able to describe the possible

worlds in some extent, hence it must be based on a finite set of symbols and rules /grammar/ that underline the formation of expressions suitable for the description of the different properties of some or other worlds. Consequently a language must contain the following components:

Language = $\langle$syntax, set of possible worlds, validity$\rangle$. In symbols $L = \langle F, M, \models \rangle$.

Syntax consists of a set of strings of symbols /texts/ and the system of rules working on these texts. Validity is a binary relation characterizing the connection between the syntactically correct expressions and the possible worlds. The set of possible worlds together with the validity relation form the semantics of the language. It follows from the role of language that the modelling of semantics is primary over that syntax. It is for this reason that the so called Model Theory is of primary importance. In order to be able to make valid statements about worlds and in order to conclude from the world models we must switch over to logic from language. Logic is comprised of the pair $\langle$ language, calculus $\rangle$. Calculus is a system of rules <u>of inference</u> that permits reasoning about the worlds.

Mathematical logic is characterized by the fact that within its framework the possible worlds are represented by some mathematical objects. These mathematical objects are called models. /They are the mathematical models of the possible worlds./

So, in mathematical logic, the set of possible worlds is represented by the set of models. The individual branches of mathematical logic differ from each other in the mathematical objects used for the representation of the worlds. E.g. classical mathematical logic defines the models as structures. The mathematical theory of semantics is called Model Theory. The construction of a calculus to a language belongs to the so called Model Theory, and the refinement of already existing calculuses belongs to the so called Proof Theory. In this work we restrict ourselves to the investigation of classical logic.

II. BASIC DEFINITIONS AND NOTATIONS

In this work we strongly rely on the tools and concepts of naive set theory. Now we introduce the notations :

iff is a shorthand for "if and only if".

w.r.t. is a shorthand for "with respect to".

▲ marks the end of proofs.

● marks the end of other units e.g. definitions, examples.

$\omega \stackrel{d}{=} \{0, 1, 2, \ldots\}$, that is ω denotes the set of integers.

$\emptyset$ is the empty set.

Two sequences $\langle a_i \rangle_{i \le n}$ and $\langle b_i \rangle_{i \le n}$ are identical iff for all $i \le n$, $a_i = b_i$.

An n-ary relation is a set of sequences /of length n/ i.e. an n-ary relation on the sets $A_1, \dots, A_n$ is a subset of $A_1 \times \dots \times A_n$.

A function is a set of pairs.

Let the function f be such that $f \subseteq A \times B$. The __range__ of f (Rgf) is the smallest set for which $f \subseteq A \times Rgf$. Similarly the __domain__ of f ($Do\,f$) is the smallest set for which $f \subseteq Dof \times B$.

$f: A \to B$ means that Do f = A and Rgf $\subseteq$ B.

$f: A \to B$ is __onto__ if Rgf = B.

An __n-ary function__ f from A to B is a __set of sequences__ of length $n+1$ i.e. $f \subseteq A \times \dots \times A \times B$ i.e. $f: {}^nA \to B$.

Thus an n-ary function is an __$n+1$ ary relation.__ This notation is justified by the fact that to make $\times$ associative we define $A_1 \times \dots \times A_n \overset{d}{=} (\dots((A_1 \times A_2) \times A_3) \times \dots) A_n$ i.e.

$$\langle a_1, a_2, \dots, a_n \rangle \overset{d}{=} \langle \dots \langle\langle a_1, a_2 \rangle, a_3 \rangle, \dots, a_n \rangle$$

Now we show how to use functions and relations.

If $f: {}^nA \to B$ then $f(a_1, \dots, a_n) = b$ means that $\langle a_1, \dots, a_n, b \rangle \in f$.

If we state that the elements $a_1, \dots, a_n$ are in relation with r where $r \subseteq A_1 \times \dots \times A_n$ then this means that $\langle a_1, \dots, a_n \rangle \in f$.

If $\langle a_1, \dots, a_n \rangle \notin r$ then the relation r does not hold between the elements $a_1, \dots, a_n$. If $a \notin Dof$ then we say that the function f is not defined on a.

A function is a special relation. The property which distinquishes a <u>function</u> from an arbitrary relation is the <u>uniqueness</u> of its values, that is $f \subseteq A \times B$ is a function iff $\langle a, b_1 \rangle$, $\langle a, b_2 \rangle \in f$ implies that $b_1 = b_2$.

Let r be an n-ary relation /or an $n-1$-ary function/. Then the relation /or function, respectively/ $r \cap {}^n B$ is the <u>restriction</u> of r to B.

Now we introduce the <u>concept of algorithms.</u>

II.1. <u>Concept of algorithms</u>

We shall use the intuitive concept of algorithm. To state our theorems it is enough to postulate some basic properties of algorithms. These postulates are satisfied by any intuitive concept of algorithm as well as by any mathematical concept of algorithm defined so far. It is left to the reader to decide which concept of algorithm to choose e.g. computer program, Turing machine, Markov algorithm, recursive functions.

We postulate that there is a finite set of symbols say Z by which each algorithm can be described. That is any algorithm is defined by a text /finite string of symbols/ from Z. Let Z^* stand for the set of all finite strings of symbols from Z. /Thus an algorithm - or the description of that algorithm - is an element of Z^* /.

Each algorithm ALG has a meaning: the execution of ALG. From this meaning we consider only the <u>partial function</u> $\overline{ALG}$ from a subset of Z^* into Z^*. For any X, Y, ALG$\in Z^*$ the statement $\overline{ALG}(X) = Y$ means that ALG is an algorithm and if we execute ALG for the input X the execution stops in a finite number of steps and the result is Y. The notations $Rg\ \overline{ALG}$ and $Do\ \overline{ALG}$ are taken in the strict sense, i.e. if $X \in Do\ \overline{ALG}$ then ALG is defined on it.

But for convenience purposes, just in the case of algorithms we sometime abuse the notation $f: A \to B$; even if $Do\ \overline{ALG} \neq Z^*$ we write $\overline{ALG} : Z^* \to Z^*$. We suppose that the set of numerals is included in Z^* and by a slight sloppiness we write $\omega \subseteq Z^*$.

We further postulate that if we have a straightforward definition of a symbol-manipulation procedure then it is an algorithm, and if we combine two or more algorithms in an intuitively straightforward way than the new object is again an algorithm.

An algorithm ENU <u>enumerates</u> a set $B \subseteq Z^*$ iff $Do\ \overline{ENU} = \omega$ and $Rg\ \overline{ENU} = B$. An algorithm DEC <u>decides</u> $B \subseteq Z^*$ iff $Do\ \overline{DEC} = Z^*$ and $Rg = \{K_0, K_1\}$ where K_0 and K_1 are two distinguished elements of Z^* and for any $X \in Z^*$: $\overline{DEC}(X) = K_1$ iff $X \in B$.

<u>Proposition II.1.</u> Let $B \subseteq Z^*$. From any decision algorithm /for B/ we can always construct an enumerating algorithm but not vica versa.

Now we introduce the <u>concept of language.</u>

II.2. Concept of language

A language L is a triple:

$L = \langle F, M, \models \rangle$ where F is the syntax /set of sentences/, $\langle M, F \rangle$ is the semantics where M is the class of models /class of possible worlds/ and $F \subseteq M \times F$ is the validity relation, i.e. if $\varphi \in F$ is a sentence and $\mathfrak{A} \in M$ is a model then $\mathfrak{A} \models \varphi$ means that φ is valid /true/ in $\mathfrak{A}$.

If $\mathfrak{A} \models \varphi$ we also say that $\mathfrak{A}$ is a model of φ.

Let $\Sigma \subseteq F$, then $\mathfrak{A} \models \Sigma$ iff for all $\varphi \in \Sigma$: $\mathfrak{A} \models \varphi$.

In this case we also say that $\mathfrak{A}$ is a model of Σ.

Let $\Sigma \subseteq F$; $\psi \in F$. ψ is a consequence of Σ /in symbols $\Sigma \models \psi$/ iff all models of Σ are also models of ψ.

A $\varphi \in F$ is a tautology of L /in symbols $\models \varphi$/ iff the set of models of φ is exactly M.

The symbol $\mathfrak{A} \not\models \varphi$ means that it is not the case that $\mathfrak{A} \models \varphi$.

A logic is a pair $\langle$ L, CAL $\rangle$ where L is a language and CAL is a calculus for L. An algorithm is a calculus for L. An algorithm is a calculus for L if to any given $\varphi \in F$ CAL enumerates a set of consequences of φ. CAL is complete for L iff it enumerates the set of all tautologies of L. CAL is adequate for L iff to any given $\varphi \in F$ CAL enumerates the set of all consequences of φ.

A logic $\langle L, CAL \rangle$ is complete /or adequate/ if CAL is complete /or adequate/ for L. Note, that if CAL is a calculus for L and to a given φ CAL enumerates the set $\Sigma \subseteq F$, then for any $\psi \in \Sigma$ also $\varphi \models \psi$.

A language <u>has implications</u> iff there is a sentential connective $\rightarrow$ such that for any $\varphi, \psi \in F, \mathfrak{M} \in M$:

$$\mathfrak{M} \models (\varphi \rightarrow \psi) \quad \text{iff} \quad (\mathfrak{M} \not\models \varphi \text{ or } \mathfrak{M} \models \psi)$$

<u>Proposition II.2.</u> Let L have implications and CAL be complete for L. Then an adequate calculus CAL' for L can be obtained from CAL in the following way:

$\overline{CAL}' : F \times \omega \rightarrow F$ such that for any $\varphi \in F$ and $n < \omega$

$$\overline{CAL}'(\varphi, n) = \begin{cases} \psi & \text{if } CAL(n) = (\varphi \rightarrow \psi) \\ \varphi & \text{otherwise} \end{cases}$$

<u>Proof.</u> The proof is immediate from the definitions

▲

III. <u>THE MODELS OF A T-TYPE LANGUAGE</u> (M^t)

In our approach every language has a fixed type t. Now we define the concept of type.

Throughout this article C_0 is a fixed symbol.

Definition III.1.

A type is a pair of functions $t = \langle t', t'' \rangle$

such that 1./ $Rg\, t' \subseteq \omega \setminus \{0\}$

2./ $Rg\, t'' \subseteq \omega$

3./ Dot' is disjoint from Dot''

4./ $\langle c_0, 0 \rangle \in t''$

$Do\, t'$ is called the set of <u>relation symbols,</u> and

$Do\, t''$ is called the set of <u>function symbols.</u>

If r is a relation symbol, then $t'(r)$ is its arity, and similarly for function symbols.

/Note that the purpose of $\langle c_0, 0 \rangle \in t''$ will be seen at the definition of t-type models /see Definition III.2./.

<u>Remark:</u> In this paper we restrict ourselves to such types t, where $Rg\, t', Rg\, t'' \subseteq \omega$. This restriction can be easily replaced by $Rg\, t', Rg\, t'' \subseteq Ord$, when Ord is the class of ordinal numbers. Though, these generalized languages are not classical in the strict sense. Such generalized languages were investigated in [1].

In this paper we restrict ourselves to the case where t' and t'' are countable, moreover we suppose the existence of algorithms $ALGT'$ and $ALGT''$ which enumerate t' and t'' respectively. That is $\overline{ALGT}': \omega \to t'$ such that for any $\langle r, n \rangle \in t'$ there is an $m < \omega$ for which $\overline{ALGT}'(m) = \langle r, n \rangle$; similarly for $ALGT''$.

A t-type model is a set on which fixed functions corresponding to the function symbols, and fixed relations corresponding to the relation symbols from t, are defined.

<u>Definition III.2.</u>

A t-type model is a function $\mathfrak{A}$ such that $\mathfrak{A}(0) \overset{d}{=} A$ is a set, and for all relation symbols r, $\mathfrak{A}(r) \subseteq {}^{t'(r)}A$ /that is $\mathfrak{A}$/r/ is a $t'(r)$-ary relation on A/, and for all function symbols f, $\mathfrak{A}(f) : {}^{t''(f)}A \to A$ and if t''/f/ = 0 then $\mathfrak{A}(f) \in A$. That is the zero-ary function symbols are the <u>constant</u> symbols /and the corresponding elements in $\mathfrak{A}$ are the constants/.

Now we define the class of t-type models M^t:

$M^t \overset{d}{=} \{\mathfrak{A} : \mathfrak{A}$ is a t-type model$\}$

Note that $\langle c_0, 0\rangle \in t''$ serves to exclude the empty model /for which $\mathfrak{A}(0) = \emptyset$/ from M^t.

<u>Notation:</u>

We denote models by German capitals, and to denote the values of a model $\mathfrak{A}$ as of function we use the following notation:

$\mathfrak{A}(0) \overset{d}{=} \mathfrak{A}_0 \overset{d}{=} A$	/the corresponding Roman capitals/
$\mathfrak{A}(r) \overset{d}{=} \mathfrak{A}_r$	for all $r \in Do\,t'$,
$\mathfrak{A}(f) \overset{d}{=} \mathfrak{A}_f$	for all $f \in Do\,t''$.

If it is not stated otherwise we suppose that a type $t = \langle t', t''\rangle$ is fixed and we call an object $\mathfrak{A}$ a model if it is a t-type model.

Sometimes we shall consider functions as special relations.

Definition III.3.

$\mathcal{B}$ is a submodel of the model $\mathfrak{A}$:

$\mathfrak{A} \supseteq \mathcal{B}$ iff for all $\langle r,n\rangle \in t'$ or $\langle r,n-1\rangle \in t''$

$$\mathcal{B}_r = \mathfrak{A}_r \cap {}^nB$$

The definition implies that $B \subseteq A$, the relations and functions are simply restricted to B, and since B has to be of type t we have, that B is closed w.r.t. the functions of $\mathfrak{A}$.

Theorem III.4.

Let $\mathfrak{A}$ be a model. The set of all submodels of $\mathfrak{A}$ is partially ordered by $\subseteq$, and this ordering has a smallest element.

Proof:

$\subseteq$'s being a partial order follows from its definition. The intersection of an arbitrary set of submodels of $\mathfrak{A}$ is again a sumbodel of $\mathfrak{A}$. This follows from the fact that all submodels contain the constants and we always have at least one constant . Now take the intersection of all submodels of $\mathfrak{A}$. This is also a submodel and of course it is the smallest one.

▲

Having defined the models we turn to define the languages treated in this work: ${}_0L^t$, ${}_1L^t$ and ${}_2L^t$. To this end we define their syntax and validity e.g. ${}_{0}F^{t}$, ${}_{0}\models^{t}$. All these three languages have the same class of models M^{t}. That is

$$ {}_{0}L^{t} = \langle {}_{0}F^{t}, M^{t}, {}_{0}\models^{t} \rangle \text{ e.t.c.} $$

IV. THE T-TYPE ZERO-ORDER LANGUAGE

IV.1. The syntax of the language:

The syntax of the languages investigated in this work have an analogous structure to that of the natural languages:

- first we construct the set of nouns. These will be called terms /T/.
- from the nouns we construct the prime sentences /P/ by relation symbols.
- from the prime sentences we construct the sentences /F/ by the sentential connectives.

Remember, that the elements of $Dot' \cup Dot''$ are symbols.

Definition IV.1.

a./ The set of t-type zero order terms ${}_{0}T^{t}$ is the smallest set /of strings of symbols/ for which:

i) if $t''(c) = 0$ then $c \in {}_{0}T^{t}$

ii) if ($t''(f) = n$ and $\tau_1, \ldots, \tau_n \in {}_{0}T^{t}$) then $f(\tau_1, \ldots, \tau_n) \in {}_{0}$

Note, that all terms are constructed from constant symbols by using function symbols. Any element τ of ${}_0T^t$ is really a noun, since in any $\mathcal{O}\!\mathcal{L} \in M^t$ to the term τ there corresponds a certain denotation, actually it denotes an element of A.

b./ The set of <u>t-type zero-order prime sentences /${}_0P^t$/</u>:

$${}_0P^t \overset{d}{=} \{ r(\tau_1, \ldots, \tau_n) : t'(r) = n \text{ and } \tau_1, \ldots, \tau_n \in {}_0T^t \}.$$

c./ The set of <u>t-type zero-order sentences /${}_0F^t$/</u>: is the smallest set /of strings/ for which the following three conditions hold:

i) ${}_0P^t \subseteq {}_0F^t$

ii) if $\varphi \in {}_0F^t$ then $\neg\varphi \in {}_0F^t$

iii) if $\varphi, \psi \in {}_0F^t$ then $(\varphi \wedge \psi) \in {}_0F^t$

Note, that the zero order sentential connectives are the symbols $\neg$ (not) and $\wedge$ (and).

Since to the set /of pairs/ t'' there is an algorithm ALGT$''$ which enumerates t'', to the set of terms ${}_0T^t$ there is an algorithm ALGTER$_0^t$ which enumerates all the terms. That is $\mathrm{ALGTER}_0^t : \omega \to {}_0T^t$ and it is onto. The construction of ALGTER$_0^t$ from ALGT$''$ is left to the reader.

Analogously we can define an algorithm ALGF$_0^t$ which enumerates the set ${}_0F^t$.

We shall introduce abbreviations to our language in the usual way, in order to make sentences more readable. The symbols $\vee$ /or/, $\rightarrow$ /implies/ and $\leftrightarrow$ /iff/ are abbreviations defined as follows:

$(\varphi \vee \psi)$ for $\neg(\neg\varphi \wedge \neg\psi)$

$(\varphi \rightarrow \psi)$ for $(\neg\varphi \vee \psi)$

$(\varphi \leftrightarrow \psi)$ for $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$

Another abbreviation which we shall adopt is to live out unnecessary parantheses.

IV.2. The semantics of the language ${}_oL^t$

We are now ready to build a "bridge" between the sentences $/{}_oF^t/$ and the models M^t. The denotation of a term $\tau \in {}_oT^t$ in a model $\mathfrak{M} \in M^t$ is $\overline{\mathfrak{M}}_\tau$. We shall express the fact that a sentence φ is valid /true/ in a model $\mathfrak{M}$ by the special notation $\mathfrak{M} \models \varphi$.

The symbol $\models$ is an abbreviation for the relation ${}_o\models^t \subseteq M^t \times {}_oF^t$.

Now we define the relation $\models \overset{d}{=} {}_o\models^t \subset M^t \times {}_oF^t$ and the denotation of the terms.

Definition IV.2.

Let c be a constant symbol i.e. $t''(c) = 0$, f a function symbol /for which $t''(f) = n$/, $\tau_1, \ldots, \tau_n \in {}_oT^t$ and r a relation symbol /for which $t'(r) = m$/.

a./ $\overline{\mathfrak{A}}_c \overset{d}{=} \mathfrak{A}_c$

$\overline{\mathfrak{A}}_{f(\tau_1,\dots,\tau_n)} \overset{d}{=} \mathfrak{A}_f(\overline{\mathfrak{A}}_{\tau_1},\dots,\overline{\mathfrak{A}}_{\tau_n})$.

b./ $\mathfrak{A} \models r(\tau_1,\dots,\tau_n)$ **iff** $\langle \overline{\mathfrak{A}}_{\tau_1},\dots,\overline{\mathfrak{A}}_{\tau_n}\rangle \in \mathfrak{A}_r$.

c./(i) $\mathfrak{A} \models \varphi \wedge \psi$ **iff** both $\mathfrak{A} \models \varphi$ and $\mathfrak{A} \models \psi$.

(ii) $\mathfrak{A} \models \neg\varphi$ **iff** $\mathfrak{A} \not\models \varphi$.

●

Example IV.3.

$t' \overset{d}{=} \{\langle r, 2\rangle\}$

$t'' \overset{d}{=} \{\langle f, 1\rangle, \langle c, 0\rangle\}$

The model $\mathfrak{A} \in M^t$ **is defined as**

$A \overset{d}{=} \{a_1, a_2, a_3\}$

$\mathfrak{A}_r \overset{d}{=} \{\langle a_1, a_2\rangle, \langle a_1, a_3\rangle\}$

$\mathfrak{A}_f(a_1) \overset{d}{=} a_2$

$\mathfrak{A}_f(a_2) \overset{d}{=} a_2$

$\mathfrak{A}_f(a_3) \overset{d}{=} a_1$

$\mathfrak{A}_c \overset{d}{=} a_1$

Now $\overline{\mathfrak{A}}_c = a_1$, $\overline{\mathfrak{A}}_{f(c)} = \overline{\mathfrak{A}}_{ff(c)} = \dots = \overline{\mathfrak{A}}_{f\dots f(c)} = a_2$

$\mathfrak{A} \models r(c, ff(c))$

$\mathfrak{A} \models \neg r(f(c), c)$

For the illustration of this example see Fig. 1.

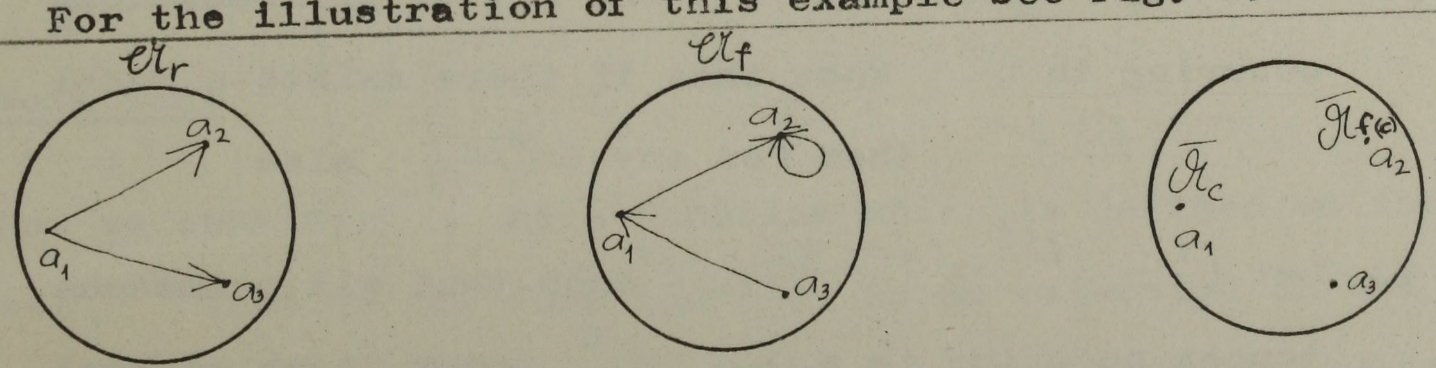

Fig. 1.

Note, that we cannot talk about a_3 since no noun refers to it. Remember, that nouns are constructed from constants according to the definition of ${}_0T^t$.

Now, we are ready to define the t-type zero-order language ${}_0L^t$:

$${}_0L^t \overset{d}{=} \langle {}_0F^t, M^t, {}_0\models^t \rangle .$$

Note, that ${}_0\models^t$ is usually denoted by $\models$.
Using the terminology of Section 2 the semantics of ${}_0L^t$ is $\langle M^t, \models \rangle$ and its syntax is ${}_0F^t$.

<u>Preparing Exercises IV.4.</u>

IV.4.1. Construct a model $\mathfrak{M}$ in which if $\tau_1 \neq \tau_2$ then $\overline{\mathfrak{M}}_{\tau_1} \neq \overline{\mathfrak{M}}_{\tau_2}$ for any $\tau_1, \tau_2 \in {}_0T^t$.

IV.4.2. Show that if Π is such that for any $\pi \in \Pi$ either $\pi \in {}_0P^t$ or there is a $\rho \in {}_0P^t$ for which $\pi = \neg\rho$ but $\rho \notin \Pi$, then there is a model $\mathfrak{M}$ such that $\mathfrak{M} \models \Pi$.

IV.4.3. Let $\sigma \in {}_0F^t$ and $\Gamma \subseteq {}_0P^t$ the set of primsentences occuring in σ. Show that if there exists a model $\mathfrak{L} \models \Gamma \cup \{\sigma\}$ then for any $\mathfrak{M} \models \Gamma$ also $\mathfrak{M} \models \sigma$.

IV.4.4. Let $\Gamma \subseteq {}_0P^t \cup \{\neg\pi\}_{\pi \in {}_0P^t}$ such that all primesentences occuring in a $\sigma \in {}_0F^t$ occur in an element of Γ as well. Prove the proposition of Exercise IV.4.3. for this case too.

V. SOME PROPERTIES OF THE ZERO-ORDER LANGUAGE ${}_0L^t$.

Recall the definition of submodel $\subseteq$ /see Definition III.3./. Now we investigate some properties of ${}_0L^t$ which are related to the notion of submodel.

Theorem V.1.

Let $\mathfrak{B} \subseteq \mathfrak{A}$ then

$\mathfrak{B} \models \varphi$ iff $\mathfrak{A} \models \varphi$

for any $\varphi \in {}_0F^t$

Proof.

The proof goes by induction on the length of the sentence φ.

For prime sentences the theorem holds by the definition of $\subseteq$, ${}_{1^0}\models^t$ and $\overline{\mathfrak{A}}_\tau$.

Suppose, that the theorem holds for the sentences ψ and χ.

Now, the theorem holds for $\psi \wedge \chi$ as well by the definition of $\mathfrak{A} \models \psi \wedge \chi$ /see Definition IV.2.c./. The theorem holds for $\neg\psi$ as well by the definition of $\neg\psi$ /see Definition IV.2.c./.

▲

Notation: $C \overset{d}{=} \{c : t''(c) = 0\}$

Now we consider $\overline{\mathfrak{A}}$ as a function which is defined on the set ${}_0T^t$. That is the function $\overline{\mathfrak{A}}$ is an extension of the function $\mathfrak{A} \cap (C \times A)$ from the set of constant symbols to the set of all nouns /or terms/.

This is illustrated on Fig. 2.

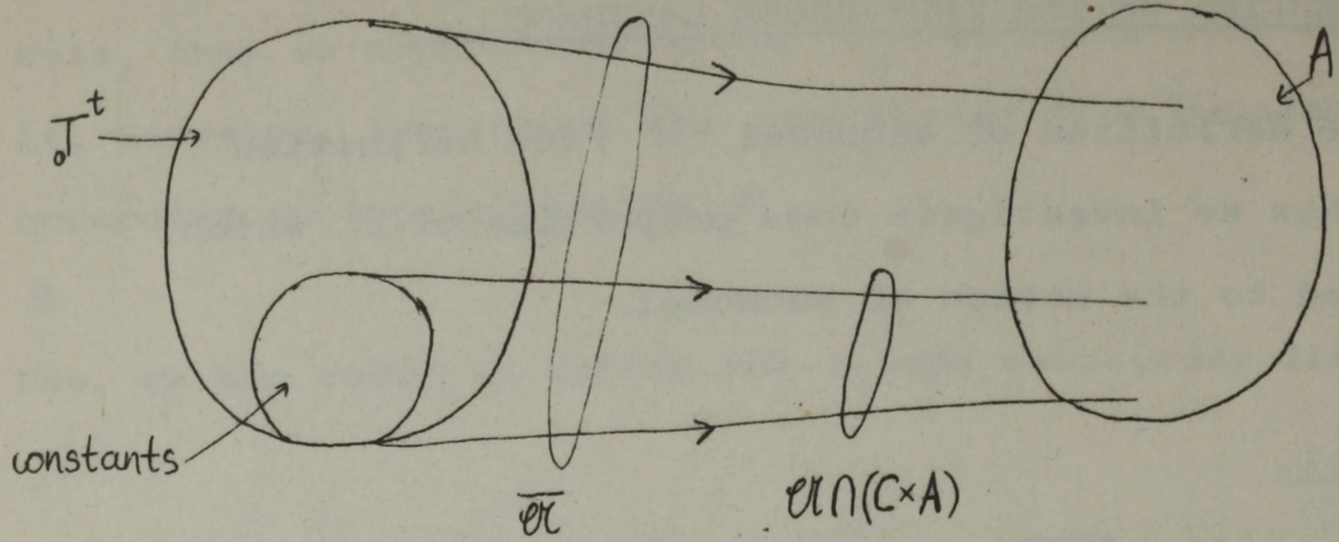

Fig. 2.

Recall that $Rg\, f$ is the range of the function f.

Theorem V.2.

Let $\mathfrak{B}$ be the smallest submodel of $\mathfrak{A}$, now

$$B = Rg\, \overline{\mathfrak{A}}$$

With other words, the smallest submodel consists of exactly those elements which have nouns referring to them /see Definition IV.1./.

Proof.

For any $\Sigma \subseteq \mathfrak{A}$ we have $Rg\, \overline{\mathfrak{A}} \subseteq C$.

In the same time $Rg\, \overline{\mathfrak{A}}$ is closed under the functions of $\mathfrak{A}$ /and thus forms a submodel of $\mathfrak{A}$/. This follows from the definition of $_0T^t$.

▲

Notation: Let $\Sigma \subseteq {_0F^t}$, now

$\mathfrak{A} \models \Sigma$ iff for any $\sigma \in \Sigma$, $\mathfrak{A} \models \sigma$.

__Lemma V.3.__ Let $\Pi \subseteq {}_0F^t$ be such that for any $\pi \in \Pi$ either $\pi \in {}_0P^t$ or there is a $\rho \in {}_0P^t$ for which $\pi = \neg\rho$ and $\rho \notin \Pi$. Then there is a model $\mathfrak{A}$ such that $\mathfrak{A} \models \Pi$.

__Proof.__

Let Π be as stated. Now, we construct a model $\mathfrak{A}$:

$\mathfrak{A}_0 = A \overset{d}{=} {}_0T^t$.

$\mathfrak{A}_f(\tau_1,\dots,\tau_n) \overset{d}{=} f(\tau_1,\dots,\tau_n)$ for any $t''(f)=n$; $\tau_1,\dots,\tau_n \in {}_0T^t$.

$\mathfrak{A}_r = \{\langle \tau_1,\dots,\tau_m\rangle : r(\tau_1,\dots,\tau_m) \in \Pi\}$, for any $t'(r)=m$.

Now, $\mathfrak{A} \models \Pi$ since $\pi \in \Pi$ implies that $\neg\pi \notin \Pi$ and $\tau_1 \neq \tau_2$ implies that $\overline{\mathfrak{A}_{\tau_1}} \neq \overline{\mathfrak{A}_{\tau_2}}$.

▲

__Definition V.4.__

The $\varphi \in {}_0F^t$ is a __tautology:__

$\models \varphi$ iff $\mathfrak{A} \models \varphi$ for all $\mathfrak{A} \in M^t$.

See also Section 2.

__Definition V.5.__

$\mathfrak{A} \models \bigvee_{i \in I} \varphi_i$ iff there is an $i \in I$ for which $\mathfrak{A} \models \varphi_i$.

$\models \bigvee_{i \in I} \varphi_i$ iff $\mathfrak{A} \models \bigvee_{i \in I} \varphi_i$ for all $\mathfrak{A} \in M^t$.

__Theorem V.6.__ /Compactness theorem of ${}_0L^t$/.

Let $\{\varphi_i\}_{i<\omega} \subseteq {}_0F^t$, then

$\models \bigvee_{i<\omega} \varphi_i$ iff there is an $n<\omega$ for which $\models \bigvee_{i<n} \varphi_i$.

__Proof.__

1./ One direction is trivial, namely if $\models \bigvee_{i<n} \varphi_i$ then $\models \bigvee_{i<\omega} \varphi_i$ as well, by Definition V.5.

2./ Let us suppose that $\models \bigvee_{i<\omega} \varphi_i$ hold. We have to prove the existence of an $n<\omega$ for which $\models \bigvee_{i<n} \varphi_i$ as well.

To this end suppose the contrary, i.e. suppose that for all $n<\omega$, $\not\models \bigvee_{i<n} \varphi_i$. From this hypothesis we have to derive $\not\models \bigvee_{i<\omega} \varphi_i$.

For all n, let $\mathfrak{A}^n$ be such that $\mathfrak{A}^n \not\models \bigvee_{i<n} \varphi_i$ we have to show the existence of a model $\mathfrak{A}^\omega$ for which $\mathfrak{A}^\omega \not\models \bigvee_{i<\omega} \varphi_i$.

Now we construct a set Π by an easy induction. $\Pi \overset{d}{=} \bigcup_{i<\omega} \Pi_i$ and now we define Π_i for all $i<\omega$. Let $_0P^t = \{\pi_0, \pi_1, \ldots, \pi_n, \ldots\}$

$\Pi_0 \overset{d}{=} \phi$

Suppose that Π_k is already defined so that for infinitely many n, $\mathfrak{A}^n \models \Pi_k$. Now, either there are infinitely many n for which $\mathfrak{A}^n \models \Pi_k \cup \{\pi_k\}$, or, if this is not the case, then there are infinitely many n for which $\mathfrak{A}^n \models \Pi_k \cup \{\neg \pi_k\}$.

In the first case $\Pi_{k+1} \overset{d}{=} \Pi_k \cup \{\pi_k\}$ and in the second case $\Pi_{k+1} \overset{d}{=} \Pi_k \cup \{\neg \pi_k\}$.

Π has a model $\mathfrak{A}^\omega$ by Lemma V.3. We show that $\mathfrak{A}^\omega \not\models \varphi_i$ for all $i<\omega$.

Let k be such that all prime sentences occuring in φ_i are "decided" by Π_k, that is, if π occures in φ_i then either $\pi \in \Pi_k$ or $\neg\pi \in \Pi_k$.

Now, there is an $n > i$ such that $\mathfrak{A}^n \models \Pi_k$ because there is only finitely many $n \leq i$, but by the construction of Π_k for infinitely many n: $\mathfrak{A}^n \models \Pi_k$.

If $n > i$, then $\mathfrak{A}^n \not\models \varphi_i$ because $\mathfrak{A}^n \not\models \bigvee_{j<n} \varphi_j$. By the construction of $\mathfrak{A}^\omega$ we have $\mathfrak{A}^\omega \models \Pi_k$ and so $\mathfrak{A}^n \not\models \varphi_i$ implies $\mathfrak{A}^\omega \not\models \varphi_i$.

▲

Now, we investigate the connection between the syntactical structure of a formula and its validity. Notice that any model $\mathfrak{A}$ defines a function $h: {}_0P^t \to \{0,1\}$ on the set of prime sentences in the following way: $h(\pi) = 1$ iff $\mathfrak{A} \models \pi$.

Definition V.7.

For any $h: {}_0P^t \longrightarrow \{0,1\}$ we define its extension $\bar{h}: {}_0F^t \to \{0,1\}$ by postulating:

(i) $\bar{h}(\pi) \overset{d}{=} h(\pi)$ if $\pi \in {}_0P^t$.

Let $\bar{h}(\varphi)$ and $\bar{h}(\psi)$ be already defined, then

(ii) $\bar{h}(\varphi \wedge \psi) \overset{d}{=} \min(\bar{h}(\varphi), \bar{h}(\psi))$

(iii) $\bar{h}(\neg\varphi) \overset{d}{=} 1 - \bar{h}(\varphi)$.

●

Notation:

Let $\Sigma \subseteq {}_0F^t$ be arbitrary.

$\bar{h}(\Sigma) = 1$ iff for all $\sigma \in \Sigma$ $\bar{h}(\sigma) = 1$.

/otherwise $\bar{h}(\Sigma) = 0$ /.

Lemma V.8.

Let $\mathfrak{A}$ and h be such that for all $\pi \in {}_0P^t$ $\mathfrak{A} \models \pi$ iff $h(\pi) = 1$. Then for any $\varphi \in {}_0F^t$, $\mathfrak{A} \models \varphi$ iff $\bar{h}(\varphi) = 1$.

Proof.

The proof is immediate from the fact that the definition of

$\bar{h}$ is similar to the definition of ${}_0\models^t$.

Lemma V.9.

For any $h : {}_0P^t \to \{0,1\}$ there exists an $\mathfrak{A}$ such that for all $\pi \in {}_0P^t$, $h(\pi)=1$ iff $\mathfrak{A} \models \pi$.

Proof:

The lemma follows from lemma V.3. Take

$$\Pi \overset{d}{=} \{\pi : h(\pi)=1\} \cup \{\neg\pi : h(\pi)=0\}$$

▲

Corollary V.1o.

Let $\Sigma \subseteq {}_0F^t$. Then, the following are equivalent :

(i) There is a model $\mathfrak{A}$ for which $\mathfrak{A} \models \Sigma$

(ii) There is a $h : {}_0P^t \to \{0,1\}$ for which $\bar{h}(\Sigma)=1$.

▲

Corollary V.11.

Let $\Sigma \subseteq {}_0F^t$. Then, the following are equivalent :

(i) $\models \Sigma$

(ii) For all $h : {}_0P^t \to \{0,1\}$, $\bar{h}(\Sigma)=1$.

▲

Note, that by the above corollaries and lemmas we established a strong connection between the language ${}_0L^t$ and the so called propositional /or sentential/ language. About the propositional language see [2].

Definition V.12.

Let $\Sigma \subseteq {}_0F^t$.

P^Σ is defined as the set of prime formulas occuring in the elements of Σ.

●

Exercises V.13.

From now on $\Sigma \subseteq {}_0F^t$.

<u>V.13.1.</u> Prove Lemmas 2, 3 and corollaries 1, 2 for the case $h : P^\Sigma \to \{0,1\}$ instead of $h : {}_0P^t \to \{0,1\}$

<u>V.13.2.</u> Show that if $\Gamma \subseteq {}_0F^t$ is finite, then there are only finitely many different functions $h : P^\Gamma \to \{0,1\}$ Note, there are exactly $2^{|P^\Gamma|}$ many different functions h.

<u>V.13.3.</u> Let $\Sigma \subseteq {}_0F^t$ be finitely satisfiable. Show that for every finite $\Gamma \subseteq \Sigma$ there is a $h : P^\Gamma \to \{0,1\}$ for which $\bar{h}(\Gamma) = 1$.

<u>V.13.4.</u> Let $\Gamma \subseteq {}_0F^t$ be finite and let H be an infinite, set of functions and $h : {}_0P^t \to \{0,1\}$ such that for every $h \in H$ there is a $\Theta \supseteq \Gamma$ for which $\bar{h}(\Theta) = 1$. Show that there is an <u>infinite</u> $H' \subseteq H$ and a $\bar{h}(\Gamma) = 1$ such that for any $h' \in H'$, $h \subseteq h'$.
/Hint: use exercise V.3.2., and conclude the infinitely many elements of H will contain exactly one function $h : P^\Gamma \to \{0,1\}$ /.

In the following exercises $\langle \Gamma_i \rangle_{i<\omega}$ is an increasing sequence of finite sets of formulas. That is Γ_i is finite and $\Gamma_j \subseteq \Gamma_i$ for all $j < i < \omega$. Furthermore we suppose that $\bigcup_{i<\omega} \Gamma_i$ is finitely satisfiable that is to each $i < \omega$ there is a h such that $\bar{h}(\Gamma_i) = 1$.

V.13.5. Let $i<\omega$ be arbitrary. Show that there is a $h_i : P^{\Gamma_i} \to \{0,1\}$ such that

(i) for infinitely many $i<j<\omega$ there is a $h_j \geq h_i$ such that $\overline{h}_j(\Gamma_j)=1$;

(ii) $\overline{h}_i(\Gamma_i)=1$.

V.13.6. Let $i<\omega$ be arbitrary, and let h_i be the function constructed in exercise V.13.5. Show that there exists a $h_{i+1} \supseteq h_i$ for which the conditions of exercise V.13.5. hold, i.e. $\overline{h}_{i+1}(\Gamma_{i+1})=1$ and for infinitely many $i+1<j<\omega$ there is a $h_j \supseteq h_{i+1}$ such that $\overline{h}_j(\Gamma_j)=1$.

V.13.7. Show that there is an increasing sequence $\langle h_i \rangle_{i<\omega}$ /that is if $j<i$ then $h_j \subseteq h_i$ / for which $\overline{h}_i(\Gamma_i)=1$ for every $i<\omega$.

V.13.8. Let $\langle h_i \rangle_{i<\omega}$ be an increasing sequence of functions for which $\overline{h}_i(\Gamma_i)=1$ for every $i<\omega$.
Show that if $h=\bigcup_{i<\omega} h_i$ then $\overline{h}(\bigcup_{i<\omega}\Gamma_i)=1$.
/Hint: use the definition of $\overline{h}(\Sigma)$ and the fact that for any $\sigma \in \bigcup_{i<\omega}\Gamma_i$ there is an $i<\omega$ such that $\sigma \in \Gamma_i$ /.

V.13.9. Prove the compactness theorem by using Corollary V.1.1. and exercises V.13.7, V.13.8.
/Hint: suppose that for all $n<\omega$ there is an $\mathfrak{A}$ for which $\mathfrak{A} \models \neg(\bigvee_{i<n}\varphi_i)$, conclude that for all n there is an $\mathfrak{A}$ for which $\mathfrak{A} \models \bigwedge_{i<n}\neg\varphi_i$. Take $\Sigma=\{\varphi_i\}_{i<\omega}$. Σ is clearly finitely satisfiable. Construct a function h for Σ using exercises V.13.7. and V.13.8.

Complete the proof by corollary 5.1.1. showing that there is an $\mathfrak{A}$ for which $\mathfrak{A} \not\models \bigvee_{i<\omega} \varphi_i$.

Preparing exercise V.14.

V.14.1. Construct by exercise V.13.2. and Corollary V.11. an algorithm which decides the set of tautologies. That is contruct an algorithm which, to any $\varphi \in {}_0F^t$ gives the result 1 if $\models \varphi$ and gives the result 0 otherwise.

VI. THE T-TYPE ZERO ORDER LOGIC

Recall /from Section 2/ that a logic is a pair $\langle L, CAL \rangle$, where L is a language and CAL is an algorithm, such that for any sentence φ of the language L CAL computes such sentences of the language which are consequences of φ.

When defining a calculus CAL to a language L we shall always take care of its soundness, i.e. we shall ensure that if ψ is not a consequence of φ then CAL would never compute ψ as a consequence of φ.

Definition VI.1.

The algorithm CAL is adequate w.r.t. L if for any sentence φ of L CAL enumerates the set of all consequences of φ. That is if ψ is a consequence of φ then ψ is obtainable from φ by CAL.

<u>Definition VI.2.</u>

The algorithm CAL is <u>complete</u> for L if all tautologies of L can be obtained by CAL without giving any sentence φ to CAL.

Now, we turn to give an adequate calculus to the language ${}_0L^t$.

First we define an algorithm ${}_0CAL^t$ which will be shown to be a complete calculus for ${}_0L^t$. Actually, we shall show more than just completeness of ${}_0CAL^t$: we shall show that ${}_0CAL^t$ decides the set of tautologies of ${}_0L^t$.

<u>Definition VI.3.</u>

Given an arbitrary $\varphi \in {}_0F^t$, the algorithm ${}_0CAL^t$:

first: computes the set of the functions

$$H \overset{d}{=} \{h : (h : \underline{P}^{\{\varphi\}} \longrightarrow \{0, 1\})\}$$

/Note that H is finite by Exercise V.13.2. and thus computable from φ/

second: computes $\bar{h}(\varphi)$ for all functions $h \in H$.

/Note that $\bar{h}$ is computable./

third: computes the minimum of $\{\bar{h}(\varphi) : h \in H\}$ and this minimum is the result of ${}_0CAL^t$.

<u>Notation:</u> Given an algorithm ALG, then the function computed by ALG is denoted by $\overline{\text{ALG}}$. That is if ALG is applied an α then the result is denoted by $\overline{\text{ALG}}(\alpha)$.

Theorem VI.4.

if $\varphi \in {}_0F^t$ is a tautology then ${}_0\overline{CAL}^t(\varphi)=1$, otherwise ${}_0\overline{CAL}^t(\varphi)=0$, i.e. the algorithm ${}_0CAL^t$ decides the set of zero-order tautologies.

Proof:

The theorem is immediate by corollary V.11. and exercise V.13.1. and V.13.2.

▲

Corollary VI.5.

/Completeness theorem of zero-order logic/:

${}_0CAL^t$ is a complete calculus for ${}_0L^t$.

▲

Exercise VI.6.

Construct an algorithm ${}_0\overline{ECAL}^t : \omega \to {}_0F^t$ which enumerates the set of zero order tautologies, that is:

$\models \varphi$ iff there is an $n < \omega$, ${}_0\overline{ECAL}^t(n) = \varphi$.

Definition VI.7.

The algorithm ${}_0ACAL^t$ computes the function ${}_0\overline{ACAL}^t : {}_0F^t \times {}_0F^t \to \{0,1\}$ such that for any $\varphi, \psi \in {}_0F^t$,

$${}_0\overline{ACAL}^t(\varphi,\psi) \stackrel{d}{=} {}_0\overline{CAL}^t((\varphi \to \psi))$$

Notation:

ψ is a consequence of Σ:

$\Sigma \models \psi$ iff for every $\mathfrak{A}$, $\mathfrak{A} \models \Sigma$ implies $\mathfrak{A} \models \psi$.

Instead of $\{\varphi\} \models \psi$ we write $\varphi \models \psi$.

Theorem VI.8.

/Adequateness theorem of zero-order logic/

${}_0ACAL^t$ is an adequate calculus to the zero-order logic. That is for any $\varphi, \psi \in {}_0F^t$,

$$\varphi \models \psi \text{ iff } {}_0\overline{ACAL}^t(\varphi, \psi) = 1 .$$

Proof.

The proof follows from the definition VI.7.

▲

Note, that the important result is formulated in Theorem VI.4. which postulates completeness and decidability of zero-order logic.

Exercises VI.9.

VI.9.1. Using ${}_0ACAL^t$ construct an algorithm ${}_0ACAL_1^t$ which for any given $\varphi \in {}_0F^t$ enumerates the set of consequences of φ. That is ${}_0\overline{ACAL}_1^t : {}_0F^t \times \omega \to {}_0F^t$ such that for any φ and any ψ such that $\varphi \models \psi$ there is an $n < \omega$ for which ${}_0\overline{ACAL}_1^t(\varphi, n) = \psi$. With other words:

$\{\psi :$ there is an $n < \omega$ for which ${}_0\overline{ACAL}_1^t(\varphi, n) = \psi\} = \{\psi : \varphi \models \psi\}$

VI.9.2. Show the adequateness of ${}_0\overline{ACAL}_1^t$.

VI.9.3. Construct a calculus ${}_0ACAL_2^t$ from ${}_0ACAL_1^t$ which to any finite $\Sigma \subseteq {}_0F^t$ enumerates the set of all consequences of Σ.

Now we introduce some calculuses which are complete in some generalized sense, i.e. calculuses which can treat infinite sets of formulas.

We say that an algorithm ALG is the definition of Σ if ALG enumerates Σ i.e. $\Sigma = \{\varphi \in {}_oF^t :$ there is a $n<\omega$ for which $\overline{ALG(n)} = \varphi\}$.

An algorithm which to any set $\Sigma \subseteq_o F^t$ enumerates the set of the consequences of Σ from the definition of Σ is called a <u>generalized calculus.</u>

<u>Definition VI.1o.</u>

The algorithm ${}_oGCAL^t$ is defined on the set of algorithms as follows:

for any given algorithm ALG

$${}_o\overline{GCAL}^t(ALG) = 1 \quad \text{iff}$$

$$(i) \quad \max\{{}_o\overline{CAL}^t(\bigvee_{i<n} \overline{ALG(i)}) : n<\omega\} = 1$$

and it is undefined otherwise.

The existence of ${}_oGCAL^t$ as an algorithm follows from the fact, that if /i/ holds then for some $n<\omega$ (ii) ${}_o\overline{CAL}^t(\bigvee_{i<n}\overline{ALG(i)}) = 1$.
Thus ${}_oGCAL^t$ can successively compute these values for $n=1,2,\dots$ until an $n<\omega$ is found for which (ii) holds. If such an n is found, then the result of ${}_oGCAL^t$ is 1 and the algorithm ${}_oGCAL^t$ stops. Otherwise it proceeds infinitely, i.e. the result is undefined.

<u>Theorem VI. 11.</u>

/Generalized completeness theorem of zero-order logic/:

For any algorithm ALG, $\models \bigvee_{n<\omega} \overline{ALG(n)}$ iff ${}_o\overline{GCAL}^t(ALG) = 1$.

Proof.

By the Compactness Theorem, (Theorem V.6.) $\models \bigvee_{n<\omega} \overline{ALG}(i)$ implies that $\models \bigvee_{i<n} \overline{ALG}(i)$ for some $n<\omega$ and thus ${}_0GCAL^t$ stops for this n. The other direction is trivial by the definition of the concept of calculus /see Section 2./.

▲

Definition VI.12.

The algorithm ${}_0AGCAL^t$ is defined on the Cartasian product of the set of algorithms and ${}_0F^t$ as follows:

For any given ALG and $\psi \in {}_0F^t$

Let $\overline{ALG}' : \omega \to {}_0F^t$ be defined as

$\overline{ALG}'(m) \stackrel{d}{=} (\bigwedge_{i<m} \overline{ALG}(i) \to \psi)$ for all $m<\omega$.

For any given ALG and $\psi \in {}_0F^t$

$\overline{{}_0AGCAL^t}(ALG, \psi) \stackrel{d}{=} {}_0GCAL^t(ALG')$

●

Theorem VI.13.

/Generalized adequatness theorem of zero-order logic./

For any algorithm ALG and $\psi \in {}_0F^t$

$\{\overline{ALG}(i) : i<\omega\} \models \psi$ iff ${}_0\overline{AGCAL}^t(ALG, \psi) = 1$

Proof.

The theorem is immediate by the Compactness Theorem /see Theorem V.6./.

▲

Note, we can also say that the generalized calculus "proves" ψ from the set of "hypotheses" $\{\overline{ALG}(n) : n<\omega\}$

Recognise, that an infinite set of hypotheses can only be given "effectively" by an algorithm /say ALG/. /Such a set of hypotheses is e.g. Peano's axiom system for arithmetic./ Thus, a logic also involves a concept of "proof" or "derivation" /w.r.t. its language/.

In the textbooks on propositional logic the generalized adequateness theorem is better known then the generalized completeness theorem. For the reader familiar with the former but unfamiliar with the latter it is very easy to show that the generalized completeness theorem /Theorem VI.11/ is an immediate consequence of the generalized adequateness theorem /Theorem VI.13./.

Exercises VI.14.

VI.14.1. Prove a Generalized completeness theorem from the Generalized adequateness theorem.

/Hint: Let $\overline{ALG}: \omega \longrightarrow {}_0F^t$ be arbitrary, and $\overline{ALG}^+(\varphi) \stackrel{d}{=} \neg \overline{ALG}(\varphi)$ for all $\varphi \in {}_0F^t$.

Now, $\models \bigvee_{i<\omega} \overline{ALG}(i)$ iff $(\bigwedge_{i<\omega} \overline{ALG}^+(i)) \models (\varphi \wedge \neg\varphi)$ iff

/By the Generalized Adequateness theorem, Theorem VI.13./ ${}_0\overline{AGCAL}^t(ALG^+, (\varphi \wedge \neg\varphi)) = 1$.

Now, define ${}_0\overline{GCAL}^t(ALG) \stackrel{d}{=}_0 \overline{AGCAL}^t(ALG^+, (\varphi \wedge \neg\varphi))$

VI.14.2. Define an algorithm $ACAL_3$ which for any $\varphi \in {}_0F^t$. enumerates the set of all consequences of φ

Hint: use $ALGF_0^t$ /see Section IV.1./.

VI.14.3. Show that $ACAL_3$ is an adequate calculus for ${}_0L^t$.

VI.14.4. Define explicitely the concept of "proof" in the case of ${}_{o}ACAL^{t}$ and $ACAL_{3}$.

VI.14.5. Define an algorithm $AGCAL_{2}$ which for any ALG, enumerates the set of all consequences of $\{ALG(n) : n<\omega\}$. Of course it is supposed that $\{ALG(n): n<\omega\} \subseteq {}_{o}F^{t}$.

Hint: proceed similarly as in the case of exercise 1, however in this case a "diagonalisation" is necessary. /This diagonalisation involves a similar technique to the simulation of parallel programs on a serial processor or to the organisation of time-shearing operating systems./

VI.14.6. Prove the adequateness of $AGCAL_{2}$.

VII. PROOF THEORY OF ZERO-ORDER LOGIC

If the reader is not interested in Proof Theory, this section can be skipped. The following sections can be entirely understood without reading this one.

To find a calculus which is complete for a language is an exercise in Model Theory. However, if a complete calculus is already given, to find new ones which satisfy certain additional requirements /effectiveness, naturalness etc./ belongs to the area of Proof Theory. The tools of Proof Theory are purely syntactical; and this makes Proof Theory entirely different from Model Theory in character as well. The main reason for this is that in Proof Theory we need not care about the meaning of the syntactical entities we just compair syntactical properties of different syntactical systems.

Now, we construct some more effective algorithms /calculuses/ which are equivalent to ${}_0CAL^t$. That is more effective algorithms which decide the set of tautologies. After this it is a trivial matter to construct adequate calculuses from these complete ones similarly to the case when ${}_0ACAL^t$ was constructed from ${}_0CAL^t$.

One of these more effective calculuses is the so called Resolution algorithm ${}_0RCAL^t$. Since we do not need it later we shall not define it precisely here, instead we illustrate it in the following example. /The precise definition can be found in any textbook on automatic theorem proving./

<u>Example VII.1.</u>

The algorithm ${}_0RCAL^t$ is defined as follows:

Let the sentence φ be given to ${}_0RCAL^t$ then:

a./ Compute the disjunctive normal form of φ ; form a matrix from this.

e.g. Let the disjunctive normal form of φ be $(a\wedge b\wedge \neg c)\vee(a\wedge\neg b\wedge\neg c)\vee(\neg a\wedge\neg c)\vee c$ where $a,b,c\in {}_0P^t$; then its matrix is:

1	a	b	$\neg c$
2	a	$\neg b$	$\neg c$
3	$\neg a$		$\neg c$
4			c

b./ Resolve two rows of the matrix, to obtain a new row, and attach it to the matrix. E.g. in the above matrix resolving rows 1 and 2 we obtain the row $(a, \neg c)$ and then we join this new row to the matrix as row 5.

5	a		$\neg c$

c./ Repeat step 2 until either we get the empty row or a deadline is reached. If the empty row is reached then $\overline{{}_0RCAL^t}(\varphi) = 1$; in the case of a deadline $\overline{{}_0RCAL^t}(\varphi) = 0$. This completes the definition of ${}_0RCAL^t$.

It is not too difficult to see that ${}_0RCAL^t$ gives the same result as ${}_0CAL^t$, that is

$$\overline{{}_0RCAL^t} = \overline{{}_0CAL}^t .$$

Exercises VII.2.

VII.2.1. Show that ${}_0RCAL^t$ is a complete calculus for ${}_0L^t$.

VII.2.2. Construct an adequate calculus from ${}_0RCAL^t$.

Now we are going to introduce some so called axiomatic calculuses.

Notation:

$S_\omega({}_0F^t)$ stands for the set of all finite subsets of ${}_0F^t$.

Definition VII.3.

An axiomatic calculus consists of:

(i) an algorithm which decides a set of tautologies Ax called the set of axioms of the calculus.

(ii) Rules of inference. These are given in the form of an algorithm which decides a subset of $S_\omega({}_0F^t) \times {}_0F^t$. This set is called the set of rules of inference.

●

Definition VII.4.

A derivation of $\psi \in {}_0F^t$ from $\Sigma \subseteq {}_0F^t$ is a finite sequence $\delta_0, \dots, \delta_i, \dots, \delta_n$ of sentences for which

(i) $\delta_n = \psi$

(ii) for all $i < n$ either $\delta_i \in \Sigma \cup Ax$ or there is a finite $\Gamma \subseteq \Sigma \cup Ax \cup \{\delta_j\}_{j<i}$ such that $\langle \Gamma, \delta_i \rangle$ is a rule of inference.

●

If such a derivation exists we say that ψ is derivable from Σ, in symbols $\Sigma \vdash \psi$.

In the following when we want to distinquish between say CAL_1 and CAL_2 we write $\overset{CAL_1}{\vdash}$ and $\overset{CAL_2}{\vdash}$ respectively instead of $\vdash$.

Now we show an easy /but not very elegant/ example of an axiomatic calculus:

Definition VII.5.

The axiomatic calculus ${}_0AXC_1^t$ is defined as follows:

(i) the set of axioms:

$$ {}_0AX_1^t = \{\varphi \in {}_0F^t \; : \; {}_0\overline{CAL}^t(\varphi) = 1\} $$

(ii) the rules of inference:

$$ MP \stackrel{d}{=} \{ \langle \{\varphi, (\varphi \to \psi)\}, \psi \rangle : \varphi, \psi \in {}_0F^t \} $$

●

The set of rules of inference MP is called Modus Poneus.

Theorem VII.6.

${}_0AXC_1^t$ is an adequate calculus for ${}_0L^t$. With other words

$$ \vdash^{AXC_1^t} = \models $$

Proof.

The theorem is immediate from the adequateness of ${}_0ACAL^t$. Actually ${}_0ACAL^t$ and ${}_0AXC_1^t$ are different definitions of the same algorithms.

▲

In the above definition we utilised the decidability of the set of zero-order tautologies.

Now, we introduce an important axiomatic calculus. The importance of this calculus is in the intuitively natural form of the definition of its set of axioms.

Definition VII.7.

The axiomatic calculus ${}_0AXC_2^t$ is defined as follows:

(i) $_0Ax_2^t \overset{d}{=} \{(\varphi \to \varphi), (\varphi \to (\psi \to \varphi)),$

$((\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))),$

$((\neg\varphi \to (\psi \wedge \neg\psi)) \to \varphi),$

$(\varphi \to (\psi \to (\varphi \wedge \psi))),$

$((\varphi \wedge \psi) \to \varphi),$

$((\varphi \wedge \psi) \to \psi), \ : \ \varphi, \psi, \chi \in {_0F^t}\}$

(ii) The set of rules of inference is MP.

Remark:

The advantage of $_0AXC_2^t$ is the fact that for any $\varphi \in {_0F^t}$ we can decide by first sight whether $\varphi \in {_0Ax_2^t}$ or not, without any computation. /It is often said that in the every-day practice a calculus of this sort is used in general./ Now we inve stigate this $_0AXC_2^t$ in details.

Lemma VII.8.

$\vdash^{AXC_2} \subseteq \models$. That is for any $\varphi, \psi \in {_0F^t}$ if $\varphi \vdash^{AXC_2} \psi$ then also $\varphi \models \psi$.

Proof.

The proof is easy by checking that if $\varphi \in {_0Ax_2^t}$ then $\models \varphi$, and that if $\langle \Gamma, \psi \rangle \in MP$ then $\Gamma \models \psi$.

▲

Note, that the lemma states that ${}_0AXC_2^t$ is a calculus for ${}_0L^t$ /see Section 2./.

From now on, in this section $\vdash$ stands for $\vdash^{AXC_2^t}_{0}$.

Notation:

$\Sigma \nvdash \varphi$ means that it is not the case that $\Sigma \vdash \varphi$.

A set $\Sigma \subseteq {}_0F^t$ is consistent if there is at least one $\varphi \in {}_0F^t$ such that $\Sigma \nvdash \varphi$.

Exercise VII.9.

Let $\psi \in {}_0F^t$ be arbitrary but fixed. Show that Σ is consistent iff $\Sigma \nvdash (\psi \wedge \neg \psi)$.

We say that Σ is "maximal consistent" iff Σ is consistent but the only consistent set of sentences which includes Σ is Σ itself.

Lemma VII.1o.

/Lindenbaum's Theorem/:

Any consistent set is contained in a maximal consistent set of sentences.

Proof.

Let Σ be consistent, and let

$$\{\varphi_0, \varphi_1, \dots, \varphi_i, \dots\}_{i<\omega} = {}_0F^t$$

We shall form an increasing chain of consistent sets of sentences:

$$\Sigma \overset{d}{=} \Sigma_0 \subseteq \Sigma_1 \subseteq \Sigma_2 \subseteq \ldots\ldots \subseteq \Sigma_n \subseteq \ldots$$

If $\Sigma_n \cup \{\varphi_n\}$ is consistent then $\Sigma_{n+1} \overset{d}{=} \Sigma_n \cup \{\varphi_n\}$, otherwise $\Sigma_{n+1} \overset{d}{=} \Sigma_n$.

Now we claim that $\Gamma \overset{d}{=} \bigcup_{n<\omega} \Sigma_n$ is consistent. Suppose not. Thus $\Gamma \vdash \varphi \wedge \neg\varphi$ but in the derivation only finitely many elements of Γ could be used, that is there is a finite $\Theta \subseteq \Gamma$ such that $\Theta \vdash \varphi \wedge \neg\varphi$. Since Θ is finite, for some $n < \omega$, $\Theta \subseteq \Sigma_n$. This means that Σ_n is inconsistent which is a contradiction.

We now claim that Γ is maximal consistent.
For suppose $\Gamma \cup \{\varphi_n\}$ is consistent, for some $\varphi_n \in F_0^t$. Then $\Sigma_n \cup \{\varphi_n\}$ is also consistent and thus $\varphi_n \in \Sigma_{n+1} \subseteq \Gamma$.

▲

<u>Corollary VII.11.</u>

For any axiomatic calculus, if $\Sigma \nvdash \varphi$ then Σ is contained in a maximal Γ for which $\Gamma \nvdash \varphi$.

<u>Proof.</u>

The proof is entirely analogous to the proof of Lindenbaum's theorem.

▲

Throughout this section we should have kept ourselves to the rule, that in Proof Theory all the arguments and statements should be syntactical, that is models $\mathfrak{A} \in M^t$ or $\models$ should not appear in the text. Eg. Completeness theorems should be proved by referring to an existing complete calculus, say ${}_0CAL^t$. Yet, in the following lemma we brake this rule, and use $\models$ simply because this proof fits better to the materials studied so far. However, a purely syntactical proof /referring to ${}_0CAL^t$ instead of $\models$ / could be obtained with the same effort.

Lemma VII.12.

/Deduction theorem/

Let $q, \rho \in {}_0F^t$ and $\Gamma \subseteq {}_0F^t$.
if $\Gamma \cup \{q\} \vdash \rho$ then $\Gamma \vdash (q \to \rho)$

Proof.

Suppose that $\Gamma \cup \{q\} \vdash \rho$

Thus there is a sequence ofsentences $\delta_1, .., \delta_i, .., \delta_n$ for which either $\delta_i \in Ax \cup \{q\} \cup \Gamma$ or there are $j, k < i$ such that $\delta_k = (\delta_j \to \delta_i)$ and $\delta_n = \rho$.

Define $\delta_i' \stackrel{d}{=} (q \to \delta_i)$

Now we show by induction on $i \leq n$ that $\Gamma \vdash \delta_i$ for all $i \leq n$. This will complete the proof since $\delta_n' = (q \to \rho)$.

Let us suppose that for any $j < i$, $\Gamma \vdash \delta_j'$.

If $\Gamma \vdash \delta_i$ then $\Gamma \vdash (q \to \delta_i)$ by the axiom $\varphi \to (\psi \to \varphi)$

If $\delta_i = q$ then $\vdash (q \to \delta_i)$ by the axiom $(\varphi \to \varphi)$.

Otherwise there are $j, k < i$ such that $\delta_k = (\delta_j \to \delta_i)$.

This means that $\vdash (\delta_k' \to (\delta_j' \to \delta_i'))$ because

$(\delta_k' \to (\delta_j' \to \delta_i')) = ((q \to (\delta_j \to \delta_i)) \to ((q \to \delta_j) \to$

and this is an axiom.

Thus $\Gamma \vdash \delta_i'$

This completes the proof.

Lemma VII.13.

Let $\Gamma \subseteq {}_0F^t$ be maximal consistent.

Then (i) $\varphi, \psi \in \Gamma$ iff $\varphi \wedge \psi \in \Gamma$.

(ii) $\varphi \in \Gamma$ iff $\neg\varphi \notin \Gamma$.

Proof:

First note that by the maximal consistency of Γ, $\Gamma \vdash \varphi$ implies that $\Gamma \ni \varphi$.

(i) $\Gamma \vdash \varphi, \psi$ iff $\Gamma \vdash \varphi \wedge \psi$ by the axioms

$(\varphi \rightarrow (\psi \rightarrow (\varphi \wedge \psi)))$,

$\varphi \wedge \psi \rightarrow \varphi$,

$\varphi \wedge \psi \rightarrow \psi$.

(ii) if $\neg\varphi \notin \Gamma$ then $\Gamma \cup \{\neg\varphi\} \vdash \psi \wedge \neg\psi$ again by the maximality of Γ and so $\Gamma \vdash \varphi$ by the deduction theorem and by the axiom $((\neg\varphi \rightarrow (\psi \wedge \neg\psi)) \rightarrow \varphi)$.

▲

Lemma VII.14.

Let $\Sigma \subseteq {}_0F^t$ be arbitrary.

Σ is consistent iff there is an $\mathfrak{A}$ for which $\mathfrak{A} \models \Sigma$.

Proof:

1./ If $\mathfrak{A} \models \Sigma$ then it is easy to see that Σ is consistent

2./ Let Σ be consistent. By Lindenbaum's theorem there is a maximal consistent $\Gamma \supseteq \Sigma$. We now construct a model of Γ. Now $\Pi \overset{d}{=} ({}_0P^t \cup \{\neg\pi : \pi \in {}_0P^t\}) \cap \Gamma$.

By Lemmas V.3. and VII.13. there is an $\mathfrak{A} \models \Pi$.

We prove by induction that also $\varphi \in \Gamma$ iff $\mathcal{H} \models \varphi$. By definition and Lemma VII.13. the statement holds for prime formulas. Let us suppose that the statement holds for ψ and φ. Then the statement holds for $\neg\psi$ since $\neg\psi \in \Gamma$ iff $\psi \notin \Gamma$ because Γ is maximal consistent. The statement also holds for $(\psi \wedge \chi)$ since $(\psi \wedge \chi) \in \Gamma$ iff $\psi, \chi \in \Gamma$ because Γ is maximal consistent.

▲

Corollary VII.15.

$\models \varphi$ iff $\{\neg\varphi\}$ is inconsistent.

Theorem VII.16.

/Completeness theorem of ${}_0AXC_2^t$/:

$$\models \varphi \quad \text{iff} \quad \vdash \varphi$$

Proof.

1./ If $\vdash \varphi$ then clearly $\models \varphi$.

2./ Suppose $\models \varphi$.

by corollary VII.15. $\{\neg\varphi\}$ is inconsistent and thus $\neg\varphi \vdash (\psi \wedge \neg\psi)$

by Lemma VII.12. $\vdash (\neg\varphi \rightarrow (\psi \wedge \neg\psi))$ and now by the axiom $((\neg\varphi \rightarrow (\psi \wedge \neg\psi)) \rightarrow \varphi)$ we conclude $\vdash \varphi$.

▲

Corollary VII.17.

/Adequateness theorem of ${}_0AXC_2^t$/:

$\models = \vdash$, that is for any $\varphi, \psi \in {}_0F^t$

$$\varphi \models \psi \quad \text{iff} \quad \varphi \vdash \psi.$$

Proof.

The corollary follows from the definition of MP that is from the inverse of the Deduction theorem /Theorem VII.12./.

▲

Exercises VII.18.

VII.18.1. Prove Corollary VII.15.

VII.18.2. Prove by a purely syntactical argument that for any $\Sigma \subseteq {}_0F^t$:

Σ is consistent iff there is a $h: P^{\Sigma} \to \{0,1\}$ for which $\bar{h}(\Sigma) = 1$

VII.18.3. Prove /by a purely syntactical argument/ that ${}_0\overline{CAL}^t(\varphi) = 1$ iff $\vdash_0^{AXC_2^t} \varphi$.

Note, that this is the completeness theorem of ${}_0AXC_2^t$ in a purely syntactical /Proof Theoretical/ form.

8. THE T-TYPE FIRST ORDER LANGUAGE

8.1 <u>The models</u> of the t-type first-order language are the t-type models, defined in Section 3.

The t-type first-order language is ${}_1L^t = \langle {}_1F^t, M^t, {}_1\models^t \rangle$, where $M^t = \{ \mathfrak{A} : \mathfrak{A}$ is a t-type model $\}$ and ${}_1F^t$ as well as ${}_1\models^t$ will be defined in the following.

8.2 <u>The syntax of the language $/{}_1F^t/$:</u>

Let V be an infinite set of symbols which we shall call the set of varables.

Of course, V should be dijoint from

$$Do\,t' \cup Do\,t'' \cup \{ \exists v, \neg, \wedge : v \in V \}.$$

<u>Definition 8.1.</u>

a/ The set of t-type first-order <u>terms</u> ${}_1T^t$: ${}_1T^t$ is the smallest set /of strings/ for which $V \subseteq {}_1T^t$ and if $(t''(f) = n$ and $\tau_1, \tau_2, \ldots, \tau_n \in {}_1T^t)$ then $f(\tau_1, \ldots, \tau_n) \in {}_1T^t$.

b/ The set of t-type first-order <u>prime-formulas</u> ${}_1P^t$:

$${}_1P^t \stackrel{d}{=} \{ r(\tau_1, \ldots, \tau_n) : r \in Do\,t' \text{ and } \tau_1, \ldots, \tau_n \in {}_1T^t \},$$

where $t'(r) = n$.

c/ The set of t-type first-order <u>formulas</u> ${}_1F^t$:

${}_1F^t$ is the smallest set /of strings/ for which the following four conditions hold:

(i) ${}_1P^t \subseteq {}_1F^t$

(ii) if $\varphi \in {}_1F^t$ then $\neg\varphi \in {}_1F^t$

(iii) if $\varphi, \psi \in {}_1F^t$ then $(\varphi \wedge \psi) \in {}_1F^t$

(iv) if $\varphi \in {}_1F^t$, $v \in V$ then $\exists v \varphi \in {}_1F^t$.

The symbol $\exists$ is called quantifier.

The symbol $\forall v$ /for all v/ is an abbreviation for $\neg \exists v \neg$.
We use also the abbreviations $\vee, \leftrightarrow, \longleftrightarrow$, defined in Section 4.1.
We suppose that the set V is such that there exists an algorithm $\overline{ALG\,V}: \omega \longrightarrow V$ which enumerates the set V. Thus, there is an algorithm $ALGF_1^t$ which enumerates the set ${}_1F^t$.

From now on T^t, P^t and F^t stand for ${}_1T^t$, ${}_1P^t$ and ${}_1F^t$ respectively.

8.3 The validity relation ${}_1\models^t$.

Given a model $\mathfrak{A}$, we shall use functions k to render concrete values from A to the variables of V. These functions k are called assignement functions.

Definition 8.2.

Let $\mathfrak{A} \in M^t$ and $k: V \longrightarrow A$.

Then $\bar{k}: T^t \longrightarrow A$ is the natural extension of k to T^t that is:

$\bar{k}(v) \overset{d}{=} k(v)$ for all $v \in V$,

$\bar{k}(f(\tau_1, \ldots, \tau_n)) \overset{d}{=} \mathfrak{A}_f(\bar{k}(\tau_1), \ldots, \bar{k}(\tau_n))$.

Now we define by recursion $\mathfrak{A}\,{}_1\!\models^t \varphi[k]$.

We write $\models$ instead of ${}_1\!\models^t$.

(i) $\mathfrak{A} \models r(\tau_1, \dots, \tau_n)[k]$ iff $\mathfrak{A}_r \ni \langle \bar{k}(\tau_1), \dots, \bar{k}(\tau_n) \rangle$

(ii) $\mathfrak{A} \models \neg\psi[k]$ iff $\mathfrak{A} \not\models \psi[k]$

(iii) $\mathfrak{A} \models (\psi \wedge \chi)[k]$ iff $\mathfrak{A} \models \psi[k]$ and $\mathfrak{A} \models \varphi[k]$.

(iv) $\mathfrak{A} \models \exists v \psi[k]$ iff there exists a k' such that $k'(w) = k(w)$ for all $v \neq w \in V$ and $\mathfrak{A} \models \psi[k']$

We say that φ is <u>valid</u> in $\mathfrak{A}$ if $\mathfrak{A} \models \varphi[k]$ for all k, that is

$$\mathfrak{A} \models \varphi \quad \text{iff} \quad (\forall k \in {}^V A)\, \mathfrak{A} \models \varphi[k].$$

Note, that ${}_0\!\models^t = {}_1\!\models^t \cap ({}_0F^t \times M^t)$ which justifies our habit of using the same symbol $\models$ for both ${}_0\!\models^t$ and ${}_1\!\models^t$.

Let $\Sigma \subseteq {}_0F^t$. $\mathfrak{A} \models \Sigma$ iff for all $\sigma \in \Sigma$, $\mathfrak{A} \models \sigma$ holds. If $\mathfrak{A} \models \Sigma$ we also say that $\mathfrak{A}$ is a model of Σ. Now, $\models \varphi$ iff $\mathfrak{A} \models \varphi$ for all $\mathfrak{A} \in M^t$. If $\models \varphi$, then φ is a tautology.

An "occurence" of v in φ is bound if it is inside of a subformula of the form $\exists v \psi$. An occurence of a variable is <u>free</u> if it is not bound.

A <u>sentence</u> is a formula, without free variables.

$${}_1S^t \overset{d}{=} \{\varphi \in {}_1F^t : \varphi \text{ contains no free variables}\}$$

Remark: if φ is a sentence, then

$$\mathfrak{A} \models \varphi \quad \text{iff} \quad \mathfrak{A} \models \varphi[k] \text{ for some k.}$$

With this, the first-order t-type language ${}_1L^t = \langle F^t, M^t, \models \rangle$ is defined.

Returning to the terminology of Section 2. the <u>semantics</u> of ${}_1L^t$ is $\langle M^t, \models \rangle$ and its syntax is F^t.

Remark:

In a certain sense, the important part of ${}_1L^t$ is the language $\langle S^t, M^t, \models \rangle$. But mainly for technical reasons we shall treat ${}_1L^t$ instead of this.

<u>Exercises 8.3.</u>

<u>8.3.1</u> Let $\varphi \in {}_1F^t$ and $x, y \in V$.

Show that a./ $\mathfrak{A} \models \exists x \varphi$ means that there is an element $x \in A$ which satisfies φ. /More precisely, given the assignement function k there is an $a \in A$ such that if $k'(x) = a$ and $k'(y) = k(y)$ for all $x \neq y \in V$, then $\mathfrak{A} \models \varphi[k']$./

b./ $\mathfrak{A} \models \forall x \varphi$ means that for all $x \in A$ the element x satisfies φ (in $\mathfrak{A}$).

c./ $\mathfrak{A} \models \exists x \forall y \varphi$ means, that there is an element $x \in A$ such that for all $y \in A$ the elements x, y satisfy φ.

d./ Show the same statements for similar examples e.g. $\forall x \exists y \forall z \varphi$.

8.3.2. Show that for any $\varphi \in {}_1F^t$ and $x \in V$:

$\mathfrak{A} \models \varphi$ iff $\mathfrak{A} \models \forall x \varphi$

Preparing exercise 8.4.

8.4.1. Let $\mathfrak{A} \in M^t$ and let $t'(r)=2$, $\mathfrak{A}_r$ be an equivalence relation on A, such that for all $\langle f, n \rangle \in t''$:

$$\mathfrak{A} \models ((r(x_1, y_1) \wedge \ldots \wedge r(x_n, y_n)) \rightarrow r(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n)))$$

and for all $\langle p, n \rangle \in t'$:

$$\mathfrak{A} \models ((r(x_1, y_1) \wedge \ldots \wedge r(x_n, y_n)) \rightarrow (p(x_1, \ldots, x_n) \leftrightarrow p(y_1, \ldots, y_n)))$$

Let $B \overset{d}{=} \{ \{ b : \langle b, a \rangle \in \mathfrak{A}_r \} : a \in A \}$

and for all $\langle p, n \rangle \in t''$ or $\langle p, n-1 \rangle \in t''$:

$\mathfrak{B}_p \overset{d}{=} \{ \langle X_1, \ldots, X_n \rangle \in {}^nB$: there are $x_i \in X_i$ $(i=1, \ldots, n)$ such that $\langle x_1, \ldots, x_n \rangle \in \mathfrak{A}_p \}$

Now, we defined the model $\mathfrak{B} \in M^t$, which is the same as $\mathfrak{A}$ except that the relation r is "changed to identity", that is if r(a,b) holds in $\mathfrak{A}$ then a and b are considered identical in $\mathfrak{B}$.

Show that for any $\varphi \in {}_1F^t$: $\mathfrak{A} \models \varphi$ iff $\mathfrak{B} \models \varphi$.

9. THE T-TYPE FIRST-ORDER LANGUAGE WITH EQUALITY

The t-type first-order language with equality is

${}_1L^t_{\doteq} \overset{d}{=} \langle {}_1F^t_{\doteq}, M^t, \models^{(\doteq)} \rangle$, where M^t is the same as in the previous sections, ${}_1F^t_{\doteq}$ and $\models^{(\doteq)}$ are defined in the following.

${}_1F^t_{\doteq} \overset{d}{=} {}_1F^{t^{\doteq}}$, where $t^{\doteq}$ is obtained from t by including the equality symbol $\doteq$ as a new binary relation symbol, i.e.:

$$t^{\perp} \overset{d}{=} \langle t' \cup \{\langle \perp, 2 \rangle\}, t'' \rangle.$$

Thus, ${}_1F^t_{\perp}$ is an extension of ${}_1F^t$ by including "equations" $\tau_1 \perp \tau_2$ into the set of prime formulas /for any $\tau_1, \tau_2 \in T^t$/.

To define validity $\left(\models^{(\perp)}\right)$ it is enough to define it for the new prime formulas:

$\mathfrak{A} \models^{(\perp)} \tau_1 \perp \tau_2 [k]$ iff $\bar{k}(\tau_1) = \bar{k}(\tau_2)$, for any $\tau_1, \tau_2 \in {}_1T^t$. The remaining part of the definition of $\models^{(\perp)}$ is the same as for ${}_1\models^t$.

A set of formulas Σ is algorithmically axiomatisable if there exists an enumerable axiom system for Σ and it is called decidably axiomatisable if there is a decidable axiom system for it, more precisely:

<u>Definition 9.1.</u>

a./ A set $\Sigma \subseteq {}_1F^t$ is algorithmically axiomatisable in ${}_1L^t$ iff there is an algorithm $\overline{AALG}: \omega \to \Sigma$ for which to any $\sigma \in \Sigma$: $\{\overline{AALG}(n) : n < \omega\} \, {}_1\models^t \sigma$.
and to any $\alpha \in \{\overline{AALG}(n) : n < \omega\}$, $\Sigma \, {}_1\models^t \alpha$.

b./ A set $\Sigma \subseteq {}_1F^t$ is decidably axiomatisable iff there is an algorithm $\overline{DALG}: {}_1F^t \to \{0, 1\}$ for which $Do\,\overline{DALG} = {}_1F^t$ and $\sigma \in \Sigma$ iff $\{\varphi \in {}_1F^t : \overline{DALG}(\varphi) = 1\} \models \sigma$.

<u>Theorem 9.2.</u>

The set of tautologies of ${}_1L^t_{\perp}$ is <u>algorithmically axiomatisable</u> in ${}_1L^t$, moreover it is decidably axiomatisable in ${}_1L^{t\perp}$ as well.

Proof

Let $x, y, z, x_1, \ldots, x_n, y_1, \ldots, y_n \in V$.

Consider the following set of axioms $(Ax_{\stackrel{\perp}{=}})$:

1./ equivalence:

a.) $x \stackrel{\perp}{=} x$

b.) $(x \stackrel{\perp}{=} y \wedge y \stackrel{\perp}{=} z) \rightarrow x \stackrel{\perp}{=} z$

c.) $x \stackrel{\perp}{=} y \rightarrow y \stackrel{\perp}{=} x$.

2./ congruence /for functions/

$$(x_1 \stackrel{\perp}{=} y_1 \wedge \ldots \wedge x_n \stackrel{\perp}{=} y_n) \rightarrow (f(x_1, \ldots, x_n) \stackrel{\perp}{=} f(y_1, \ldots, y_n)),$$

for all $\langle f, n \rangle \in t''$.

3./ for relations:

$$(x_1 \stackrel{\perp}{=} y_1 \wedge \ldots \wedge x_n \stackrel{\perp}{=} y_n) \rightarrow (r(x_1, \ldots, x_n) \leftrightarrow r(y_1, \ldots, y_n))$$

for all $\langle r, n \rangle \in t'$.

Now, it is not too difficult to prove that

$$\models^{(\perp)} \varphi \quad \text{iff} \quad Ax_{\stackrel{\perp}{=}} \models \varphi.$$

which just means that $Ax_{\stackrel{\perp}{=}}$ is an axiom system for the tautologies of ${}_1L^t_{\stackrel{\perp}{=}}$ /and of course it is decidable/. See also preparing exercise 8.4.

▲

Note, that if t is finite, then $Ax_{\stackrel{\perp}{=}}$ is also finite!

According to the above theorem the first-order language with equality ${}_1L^t_{\stackrel{\perp}{=}}$ and the first-order logic with equality can be considered as a special case of the language without equality ${}_1L^t$ and the corresponding logic. More precisely the logic with equality is just the logic without equality together with a decidable set of axioms. So there is no point in studying separately. Thus it is enough to study ${}_1L^t$ and to add $Ax_{\underline{1}}$ later if we want to apply the results to ${}_1L^t_{\underline{1}}$. Proving properties of ${}_1L^t_{\underline{1}}$ directly would serve nothing but to obscure the essence of the proofs by the presence of $Ax_{\underline{1}}$. All the basic properties and constructs can be reached in a more natural and straight-forward way by studying "pure" first-order language i.e. ${}_1L^t$ instead of a "mixed" version of it e.g. ${}_1L^t_{\underline{1}}$.

For this reason in the following we restrict ourselves to the investigation of ${}_1L^t$ /the results can always be applied to ${}_1L^t_{\underline{1}}$ using $Ax_{\underline{1}}$ /.

1o. THE T-TYPE SECOND-ORDER LANGUAGE ${}_2L^t$:

The t-type second-order language is

${}_2L^t \overset{d}{=} \langle {}_2F^t, M^t, {}_2\models^t \rangle$, where

the class of models M^t is the same as in the previous sections, ${}_2F^t$ and ${}_2\models^t$ are defined in the following.

1o.1 The syntax of the language ${}_2L^t$

The syntax ${}_2F^t$ is defined as follows:

First we fix infinite disjoint sets /of symbols/ which are disjoint from any other used set /for example they are disjoint from $Do\,t''$/.

V the set of first-order variables

V_n^F the set of n-ary function-variables, for each $n<\omega$

V_n^R the set of n-ary relation-variables, for each $n<\omega$

Definition lo.1.

a/ The second-order terms :

${}_2T^t$ is the smallest set for which

(i) $V \subseteq {}_2T^t$

(ii) if $(f \in V_n^F$ or $t''(f)=n)$ and $\tau_1, \dots \tau_n \in {}_2T^t$

then $f(\tau_1, \dots, \tau_n) \in {}_2T^t$.

b/ The second-order prime formulas :

$${}_2P^t \overset{d}{=} \{ r(\tau_1, \dots, \tau_n) \; : \; \tau_1, \dots \tau_n \in {}_2T^t \text{ and } (t'(r)=n \text{ or } r \in V_n^R) \} .$$

c/ The second-order formulas: ${}_2F^t$ is the smallest set for which

(i) ${}_2P^t \subseteq {}_2F^t$

(ii) if $(\varphi \in {}_2F^t$ and $x \in (\bigcup_{n \in \omega} (V_n^F \cup V_n^R) \cup V)$, then $\exists x \varphi \in {}_2F^t$

(iii) if $\varphi, \psi \in {}_2F^t$ then $(\varphi \wedge \psi) \in {}_2F^t$

(iv) if $\varphi \in {}_2F^t$ then $\neg \varphi \in {}_2F^t$.

lo.2. The validity relation ${}_2\models^t \subseteq (M^t \times {}_2F^t)$

Given a model $\mathfrak{A}$ we define the set of second order assignment functions as:

$${}_2K \overset{d}{=} \{ k \; : \; Do\, k = \bigcup_{n \in \omega} (V_n^F \cup V_n^R) \cup V, \text{ and if } v \in V \text{ then } k(v) \in A,$$
$$\text{if } f \in V_n^F \text{ then } k(f) \in {}^{({}^nA)}A,$$
$$\text{and if } r \in V_n^R \text{ then } k(r) \subseteq {}^nA \; \} .$$

That is the functions $k \in {}_2K$ render concrete elements to the first-order variable, concrete functions to the function variables and concrete relations to the relation variables.

Now we proceed analogously to the definition of first-order validity.

Definition 1o.2.

First given $k \in {}_2K$ we define the extension $\bar{k}$ of k to the set ${}_2T^t$:

$\bar{k}(v) \overset{d}{=} k(v)$ for any $v \in V$.

$$\bar{k}(f(\tau_1,\dots,\tau_n)) \overset{d}{=} \begin{cases} k(f)(\bar{k}(\tau_1),\dots,\bar{k}(\tau_n)), & \text{if } f \in V_n^F \\ \mathfrak{A}_f(\bar{k}(\tau_1),\dots,\bar{k}(\tau_n)), & \text{if } \langle f,n\rangle \in t'' \end{cases}$$

Now we define by recursion $\mathfrak{A}\ {}_2\models^t \varphi[k]$. We write $\models$ instead of ${}_2\models^t$.

(i) $\mathfrak{A} \models r(\tau_1,\dots,\tau_n)[k]$ iff $\begin{cases} \langle \bar{k}(\tau_1),\dots,\bar{k}(\tau_n)\rangle \in k(r), & \text{if } r \in V_n^R \\ \langle \bar{k}(\tau_1),\dots,\bar{k}(\tau_n)\rangle \in \mathfrak{A}_r, & \text{if } \langle r,n\rangle \in t' \end{cases}$

(ii) $\mathfrak{A} \models \neg\varphi[k]$ iff $\mathfrak{A} \not\models \varphi[k]$

(iii) $\mathfrak{A} \models (\varphi \wedge \psi)[k]$ iff $(\mathfrak{A} \models \varphi[k]$ and $\mathfrak{A} \models \psi[k])$

(iv) for any $z \in \bigcup_{n<\omega}(V_n^F \cup V_n^R) \cup V$:

$\mathfrak{A} \models \exists z \psi[k]$ iff there exists a $k' \in {}_2K$ such that $k'(w) = k(w)$ for all $z \neq w \in (\bigcup_{n<\omega}(V_n^F \cup V_n^R) \cup V)$ and $\mathfrak{A} \models \psi[k']$.

Now, the definition of second-order t-type language is complete. The free variables and sentences are defined analogously as in the first-order case.

Note, that using the same symbol $\models$ for the second-order, first-order and zero-order validity relation is completely justified by the fact, that, denoting the i-th order relation by ${}_i\models$, if $i \leq j$, then ${}_i\models = (M^t \times {}_iF^t) \cap {}_j\models$. That is, they are practically the same relation.

With other words, ${}_0F^t \subseteq {}_1F^t \subseteq {}_2F^t$, ${}_0M^t = {}_1M^t = {}_2M^t = M^t$ and ${}_0\models^t = {}_1\models^t = {}_2\models^t$.

Thus ${}_0L^t$ is simply the "restriction" of ${}_1L^t$ to ${}_0F^t$ and ${}_1L^t$ is the restriction of ${}_2L^t$ to ${}_1F^t$.

Now we introduce the following abbreviation for ${}_2L^t$ too:

$\forall v$ is an abbreviation for $\neg\exists v\neg$

/for any $v \in (\bigcup_{n<\omega}(V_n^F \cup V_n^R) \cup V)$/.

<u>Exercises 1o.3.</u>

<u>1o.3.1.</u> Let $\varphi \in {}_2F^t$ and $f \in V_n^F$.

Show that $\mathfrak{A} \models \exists f \varphi$ means that there is a function $f : {}^nA \to A$ which satisfies φ /in $\mathfrak{A}$ /. More precisely given $k \in {}_2K$ there is a function $g : {}^nA \to A$ such that if $k'(f) = g$ and $k'(v) = k(v)$ for all the other variables, then $\mathfrak{A} \models \varphi[k]$.

<u>1o.3.2.</u> Construct the parallel of exercise 8.3.1. for ${}_2F^t$ and $v_1, v_2 \in \bigcup_{n<\omega}(V_n^F \cup V_n^R) \cup V$.

<u>1o.3.3.</u> Show that ${}_0F^t \subseteq {}_1F^t \subseteq {}_2F^t$ and

${}_0\models^t = {}_1\models^t \cap (M^t \times {}_0F^t)$

${}_1\models^t = {}_2\models^t \cap (M^t \times {}_1F^t)$.

/Hint: show that if $\varphi \in {}_0F^t$ then also $\varphi \in {}_1F^t$ and $\mathfrak{A} \,{}_0\models^t \varphi$ iff $\mathfrak{A} \,{}_1\models^t \varphi$. similarly for $\varphi \in {}_1F^t$, ten also for $\varphi \in {}_2F^t$ etc./.

1o.3.4. Let $\varphi, \psi \in {}_2F^t$ and $\models (\varphi \leftrightarrow \psi)$.
Show that for any $\alpha\varphi\beta \in {}_2F^t$:
$\models (\alpha\varphi\beta \leftrightarrow \alpha\psi\beta)$,
where $\alpha\varphi\beta$ is a text which is obtained from the texts α, φ and β by concatenating them, (e.g. if $\alpha = bb$, $\varphi = pu$ and $\beta = ul$ then $\alpha\varphi\beta = bbpuul$).

1o.3.5. Show that for any $\varphi \in {}_2F^t$ and any $v \in \bigcup_{n<\omega} (V_n^F \cup V_n^R) \cup V$,
$\mathfrak{A} \models \varphi$ iff $\mathfrak{A} \models \forall v \varphi$.

1o.3.6. Show that if $\varphi \in {}_1F^t$ and $\{x_1, \dots, x_n\}$ is the set of free varables of φ then $\forall x_1, \dots, x_n \varphi \in {}_1S^t$ and $\mathfrak{A} \models \varphi$ iff $\mathfrak{A} \models \forall x_1, \dots, x_n \varphi$.

1o.3.7. Let $\mathfrak{B} \subseteq \mathfrak{A}$, $\exists x \psi \in {}_1S^t$ and ψ be quantifier-free. Show that

a) if $\mathfrak{B} \models \exists x \psi$ then also $\mathfrak{A} \models \exists x \psi$.

b) $\mathfrak{A} \models \exists x \psi$ does not imply $\mathfrak{B} \models \exists x \psi$.

c.) all conditions are necessary to prove a.)

1o.3.8. Let $\mathfrak{B} \subseteq \mathfrak{A}$ and $\varphi \in {}_1F^t$. Show that neither $\mathfrak{B} \models \varphi$ implies $\mathfrak{A} \models \varphi$ nor vice versa.

1o.3.9. Let $\exists x \forall y \psi \in {}_1F^t$, $\exists x$ not occure in ψ and $f \in V_1^F$. Let $\psi[y/\tau]$ be the formula obtained from ψ by replacing each occurance of y by the term τ.
Show that for any $\mathfrak{A} \in M^t$:
$\mathfrak{A} \models \exists x \forall y \psi$ iff $\mathfrak{A} \models \forall f \exists x \psi[y/f(x)]$.

/Hint: a/ If there is an $x \in A$ for which all $y \in A$ satisfie φ then to this $x \in A$ also $f(x) \in A$ will satisfy φ as y /by any choice of f./.

b/ If there is no $x \in A$ for which all $y \in A$ would satisfy ψ then there is a function $f: A \to A$ such that to any $x \in A$, $f(x) = y$ does not satisfy φ /.

11. SOME PROPERTIES OF THE LANGUAGES INTRODUCED SO FAR

Now we introduce the following notation:

Let $\varphi \in F_1^t$, $v \in \bigcup_{n<\omega} V_n^F \cup V_n^R \cup V$ and $\tau \in T^t$.

$\varphi[v/\tau]$ denotes the formula obtained from φ by replacing each free occurence of v in φ by τ and of course renaming bound variables to avoid collisions.

More precisely:

$r(\tau_1, \ldots, \tau_n)[v/\tau]$ is defined as above /there is no collision/.

$$(\neg\psi)[v/\tau] \overset{d}{=} \neg(\psi[v/\tau])$$

$$(\psi \wedge \chi)[v/\tau] \overset{d}{=} (\psi[v/\tau] \wedge \chi[v/\tau])$$

if w does not occure in τ, then

$$(\exists w\, \psi)[v/\tau] \overset{d}{=} (\exists w)(\psi[v/\tau])$$

otherwise, let $z \in V$ be a totally new variable,

$$(\exists w\, \psi)[v/\tau] \overset{d}{=} (\exists z\, (\psi[w/z]))[v/\tau]$$

Similarly, $\varphi[v_1/\tau_1, \ldots, v_n/\tau_n]$ denotes the formula obtained from φ by replacing each free occurrence of v_1 in φ by τ_1, ... and each free occurrence of v_n in φ by τ_n /and of course renaming bound variables if necessary/.

Definition 11.1.

$\varphi \models \psi$ iff $(\forall \mathfrak{A})(\mathfrak{A} \models \varphi \Rightarrow \mathfrak{A} \models \psi)$

$\varphi \equiv \psi$ iff $(\varphi \models \psi$ and $\psi \models \varphi)$

A formula is called quantifier-free iff it contains no quantifier, i.e. it does not contain the symbol $\exists$.

Definition 11.2.

The formula φ is in prenex normal form, iff

$\varphi = \exists x_1 \forall y_1 \ldots \exists x_n \forall y_n \psi$

where ψ is quantifier-free.

Theorem 11.3.

To every first-order formula φ there exists a $\varphi' \equiv \varphi$ which is in prenex normal form.

Proof

The proof goes by induction on the length of φ and is based on the following equivalences:

$$\neg \exists x \varphi \equiv \forall x \neg \varphi$$
$$\neg \forall x \varphi \equiv \exists x \neg \varphi$$

Let $z \in V$ not occure in $\varphi \wedge \psi$.

Then: $(\exists x \psi) \wedge \chi \equiv (\exists z \psi[x/z]) \wedge \chi \equiv \exists z (\psi[x/z] \wedge \chi)$

$(\forall x \psi) \wedge \chi \equiv (\forall z \psi[x/z]) \wedge \chi \equiv \forall z (\psi[x/z] \wedge \chi)$

$\varphi \wedge \psi \equiv \psi \wedge \varphi$.

These equivalences can easily be seen to hold by the definition of $\models$. To complete the proof see exercise 8.3.2.

Now, the above equivalences can be used mechanically to obtain a "prenex normal form" of a formula. This gives birth to the following algorithm.

Definition 11.4.

The algorithm $\overline{PREN}\ {}_1F^t \rightarrow {}_1F^t$ is defined as: for any $\varphi \in {}_1F^t$, $\overline{PREN}(\varphi) \equiv \varphi$ and $\overline{PREN}(\varphi)$ is in prenex normal form.

●

The existence of the algorithm PREN follows from the proof of the above theorem.

Notation:

$\exists x_1 \exists x_2 \ldots \exists x_n \stackrel{d}{=} \exists x_1 \ldots x_n$ and similarly for the symbol $\forall$.

Theorem 11.5. /Skolem's normal form Theorem/.

$$\exists x_1 \forall y_1 \ldots \exists x_n \forall y_n \psi \equiv \forall f_1 \ldots f_n \exists x_1 \ldots x_n \psi[y_1/f_1(x_1), \ldots, y_n/f_n(x_1, \ldots, x_n)]$$

where $\psi \in {}_1F^t$, $x_1, \ldots, x_n, y_1, \ldots, y_n \in V$ and $f_i \in V_i^F$ for all $i < n$ and $f_1, \ldots, f_n$ do not occure in ψ

Proof

It is enough to show that $\exists x_1 \ldots x_m \forall y_m \chi \equiv \forall f_m \exists x_1 \ldots x_m \chi[y_m/f_m(x_1, \ldots$ for any $\chi \in {}_1F^t$ /note that χ might contain quantifiers/. If we have shown this, the proof can be completed by an easy induction.

The statement to be proved follows from the definition of in the following way:

We give the proof here for $\exists x \forall y \chi$; the generalisation for the case of $\exists x_1 \dots x_m \forall y \chi$ is trivial.
If $\mathfrak{A} \models \exists x \forall y \chi$, then there exists an element $k(x) \in A$ for which any $k(y) \in A$ will satisfy χ. Thus, $\overline{k}(f(x)) \in A$ will satisfy χ as well, i.e. $\mathfrak{A} \models \forall f \exists x \chi[y/f(x)]$.
Otherwise, if $\mathfrak{A} \not\models \exists x \forall y \chi$ then there is no $k(x)$ for which all $k(y)$ is "good". This defines a function which renders to each $k(x)$ the corresponding "bad" $k(y)$.
Thus, $\mathfrak{A} \not\models \forall f \exists x \chi[y/f(x)]$.
▲

Definition 11.6.

We say that a <u>type t has enough function symbols</u> iff t contains infinitely many function symbols of each arity, i.e. for any n, there are infinitely many symbols $f_n^1, f_n^2, \dots$ such that for all $i < \omega$ $t''(f_n^i) = n$. From now on we suppose that the type t has enough function symbols.

Corollary 11.7.

If the type t has enough function symbols then to any formula $\varphi \in F^t$ there are $\tau_1, \dots, \tau_n \in T_1^t$ such that $\models \exists x_1 \forall y_1 \dots \exists x_n \forall y_n \varphi$ iff $\models \exists x_1 \dots x_n \varphi[y_1/\tau_1, \dots, y_n/\tau_n]$.

Proof

From the above theorem

$\models \exists x_1 \forall y_1 \dots \exists x_n \forall y_n \varphi$ iff $\models \forall f_1 \dots f_n \exists x_1 \dots x_n \varphi[y_1/f_1(x_1), \dots]$.

Now if we choose to each function variable f_i a corresponding function symbol f_i from F^t such that $t''(f_i) = i$ and f_i does not occure in φ, then by the definition of $\models$ and

$\vdash \forall f_1 \dots f_n \exists x_1 \dots x_n \varphi [y_1 / f_1(x_1), \dots, y_n / f_n(x_1, \dots, x_n)]$ iff

$\vDash \exists x_1 \dots x_n \varphi [y_1 / f_1'(x_1), \dots, y_n / f_n'(x_1, \dots, x_n)]$.

▲

Definition 11.8.

A formula φ is an existential formula if $\varphi = \exists x_1 \dots x_n \psi$, where ψ is quantifier-free.

Corollary 11.9.

If the type t has enough function symbols, then there exists an algorithm $\overline{EXI} : {}_1F^t \to {}_1F^t$, such that for any $\varphi \in {}_1F^t$:

$$\vDash \varphi \quad \text{iff} \quad \vDash \overline{EXI}(\varphi),$$

and $\overline{EXI}(\varphi)$ is an existential formula.

Proof

From Definition 11.4,

$\varphi \equiv \overline{PREN}(\varphi)$. By the above corollary there exists an existential formula $\overline{EXI}(\varphi) \equiv \overline{PREN}(\varphi)$.

▲

Recall, that ${}_1S^t \subseteq {}_1F^t$ is the set of t-type first-order sentences.

Theorem 11.10.

If the type t has enough function symbols, then there is an algorithm $\overline{ZER} : {}_1S^t \times \omega \to {}_0F^t$ such that for any $\varphi \in {}_1S^t$:

$$\vDash \varphi \iff \vDash \left(\bigvee_{i < \omega} \overline{ZER}(\varphi, i) \right).$$

Proof

Let $\overline{EXI}(\varphi) = \exists x_1 \dots x_n \varphi'$, where φ' is quantifier-free.

We give the proof for $\exists x \varphi$ without loss of generality.

Let $\models \varphi$ and so $\models \exists x \varphi$.

For every $\mathfrak{A} \in M^t$ by Theorem 3.4. there is a smallest submodel $\mathfrak{B}$ of $\mathfrak{A}$. Since $\models \exists x \varphi$, also $\mathfrak{B} \models \exists x \varphi$. Thus by Theorem 5.2. there is a $\tau \in_0 T^t$ for which $\mathfrak{B} \models \varphi[x/\tau]$ and so, since $\varphi[x/\tau] \in_0 F^t$ and $\mathfrak{B} \subseteq \mathfrak{A}$ we have $\mathfrak{A} \models \varphi[x/\tau]$. /by Theorem 5.1/.

Therefore $\models \bigvee_{\tau \in_0 T^t} \varphi[x/\tau]$.

In the other direction:

$\models \bigvee_{\tau \in_0 T^t} \varphi[x/\tau]$ trivially implies $\models \exists x \varphi$

Now, define $ZER(\varphi, n) \overset{d}{=} \varphi[x/ALGTER^t(n)]$. Since EXT is an algorithm ZER is also an algorithm.

▲

Note, that the above theorem /Theorem 11.1o./ has nothing to do with Herbrand's Theorem 13.15, because (i) Herbrand's Theorem is a purely syntactical /Proof Theoretical/ statement while the above theorem is strongly semantical; (ii) Herbrand's theorem states the existence of an $n < \omega$ /for which $\vdash^{AXC} \varphi$ iff $\vdash^{AXC} (\bigvee_{i<n} ZER(\varphi, i))$ while the above theorem states nothing of this sort.

Notation:

For any $\Sigma \subseteq_0 F^t$, $\models \bigvee \Sigma$ iff for any $\mathfrak{A} \in M^t$ there is a $\delta \in \Sigma$ such that $\mathfrak{A} \models \delta$.

Corollary 11.11.

Let the type t have enough function symbols.

Let $\mathcal{AL}$ be the set of algorithms. There exists an algorithm $\overline{ZE}: {}_1S^t \to \mathcal{AL}$ such that for any $\varphi \in {}_1S^t$ the algorithm $\overline{ZE}(\varphi): {}_0F^t \to \{0,1\}$ has the property:

$$\vdash \bigvee\{\psi \in {}_0F^t : \overline{ZE}(\varphi)(\psi) = 1\} \quad \text{iff} \quad \models \varphi .$$

Proof

We define $\overline{ZE}(\varphi)(\psi) = 1$ iff there is $n \in \omega$ for which $\overline{ZR}(\varphi, n) = \psi$ otherwise $\overline{ZE}(\varphi)(\psi) = 0$. The existence of the decision algorithm $\overline{ZE}$ follows from the fact that any ψ contains only finitely many terms.

▲

Remark:

By the above corollary the algorithm $\overline{ZE}$ is similar to the algorithms $\overline{PRF}$ and $\overline{EXI}$. To any first order formula φ the algorithm $\overline{ZE}$ correlates a decidable set ϕ of zero-order formulas such that $\models \varphi$ iff $\vdash \bigvee\phi$. More precisely to any $\varphi \in {}_1S^t$, $\overline{ZE}$ computes the definition of ϕ. This definition of ϕ is an algorithm $\overline{ZE}(\varphi)$ which decides the set ϕ. Recognise, that the existence of a decision algorithm is a stronger result than the existence of an enumerating algorithm. That is from $\overline{ZE}$ it is easy to construct a $\overline{ZE}_1$ such that $\overline{ZE}_1(\varphi)$ enumerates the set ϕ.

Now, we are ready to construct a complete calculus for the first-order language. To any first-order sentence φ by $\overline{ZE}$ we can compute the definition of ϕ and by the generalised completeness theorem of zero-order logic we can prove

$$\vdash \bigvee\phi \quad \text{iff} \quad \models \varphi .$$

12. THE T-TYPE FIRST-ORDER LOGIC

Now, we define an algorithm ${}_1CAL^t$ which proves the first-order tautologies by reducing a first-order sentence to a set of zero-order sentences. For any sentence $\varphi \in {}_1S^t$ the corresponding zero-order set is $\{\overline{ZER}(\varphi, n) : n < \omega\}$, the definition of which is the algorithm $\overline{ZE}(\varphi)$.

In the following we define a calculus ${}_1CAL^t$ for the first-order language by using the generalised calculus ${}_0GCAL^t$ of the zero-order logic and the algorithm $\overline{ZE}_1 : {}_1S^t \to \mathcal{R}\ell$ in the following way: for any $\varphi \in {}_1S^t$

$$ {}_1\overline{CAL}^t(\varphi) = {}_0\overline{GCAL}^t(\overline{ZE}_1(\varphi)) . $$

Notice, that the use of ZE_1 instead of ZE means that we do not use the result of Corollary 11.11 in the construction of complete first-order calculuses since ${}_0GCAL$ is complete w.r.t. enumerable sets of formulas as well.
To make the following definition simpler we do not use there the algorithm ZE_1.

Definition 12.1.
To any $\varphi \in {}_1S^t$ we define the algorithm $\overline{\varphi ZER} : \omega \to {}_0S^t$, such that for any $n < \omega$:

$$ \varphi ZER(n) = \overline{ZER}(\varphi, n) . $$

/Note, that by the terminology of the remark following Corollary 11.11. the algorithm $\varphi ZER = ZE_1(\varphi)$ /
Now, we define the algorithm ${}_1CAL^t : {}_1S^t \to \{0,1\}$ as follows:

for any $\varphi \in {}_1S^t$:

$${}_1\overline{CAL}^t(\varphi) \overset{d}{=} {}_0\overline{GCAL}^t(\varphi^{ZER}).$$

(see Definition 6.1o.)

Theorem 12.2. /Completeness theorem of first-order logic/

${}_1\overline{CAL}^t$ is a complete calculus for $\langle {}_1S^t, M^t, {}_1\models \rangle$ i.e.

for any $\varphi \in {}_1S^t$:

$$\models \varphi \quad \text{iff} \quad {}_1\overline{CAL}^t(\varphi) = 1.$$

Proof

The proof is immediate by the definition of ${}_1\overline{CAL}^t$ and the Generalised Completeness Theorem of zero-order logic /Theorem 6.11./.

▲

Corollary 12.3.

There exists an algorithm ${}_1\overline{CALF}^t : {}_1F^t \to \{0,1\}$ which is a complete calculus for ${}_1L^t$.

Proof

Let $\varphi \in {}_1F^t$ and the set of free variables of φ be $\{x_1, \dots, x_n\}$. Now, define ${}_1\overline{CALF}^t(\varphi) \overset{d}{=} {}_1\overline{CAL}^t(\forall x_1 \dots x_n \varphi)$

${}_1\overline{CALF}^t$ is complete for ${}_1L^t$ by the completeness of ${}_1\overline{CAL}^t$ for $\langle {}_1S^t, M^t, \models \rangle$

▲

Exercises 12.4.

12.4.1. Construct an algorithm ${}_1\overline{ACAL}^t : {}_1F^t \times {}_1F^t \to \{0,1\}$

such that for all $\varphi, \psi \in {}_1F^t$

${}_1\overline{ACAL}^t(\varphi, \psi) = 1$ iff $\varphi \models \psi$.

Hint: similarly to corollary 11.3. let $\{x_1, \dots, x_n\}$ be the set of free variables of φ and ψ. Now define

$${}_1ACAL^t(\varphi,\psi) = {}_1CAL^t(\forall x_1 \dots x_n \varphi \to \forall x_1 \dots x_n \psi).$$

12.4.2. Show that ${}_1ACAL^t$ is an adequate calculus for ${}_1L^t$, that is $\langle {}_1L^t, {}_2ACAL^t \rangle$ is an adequate logic.

Hint: see the Adequateness theorem of zero-order logic. /Theorem 6.8./

12.4.3. Construct an algorithm ${}_1ECAL^t: \omega \to {}_1F^t$ which enumerates the set of first-order tautologies, that is for any $\varphi \in {}_1F^t$:

$\vdash \varphi$ iff there is an $n \in \omega$, ${}_1ECAL^t(n) = \varphi$.

12.4.4. Show that $\langle {}_1L^t, {}_1ECAL^t \rangle$ is a complete logic.

12.4.5. Construct an algorithm ${}_1ACAL_3^t$ which for any $\varphi \in {}_1F^t$ enumerates the set of all consequences of φ.

/Hint: see Exercise 6.14.2./

12.4.6. Show that $\langle {}_1L^t, {}_1ACAL_3^t \rangle$ is an adequate logic, or with other words show that ${}_1ACAL_3^t$ is an adequate calculus for ${}_1L^t$.

12.4.7. ${}_0L_{qf}^t = \langle {}_0F_{qf}^t, M_1^t, \models^t \rangle$ where ${}_0F_{qf}^t$ is the set of quantifier-free first-order formulas.

a/ Give a complete calculus to ${}_0L_{qf}^t$ by the completeness theorem f ${}_0L^t$ /Corollary 6.5./.

b/ Give an adequate calculus to ${}_{0}L^{t}_{v}$ by Theorems 6.11 and 11.1o.

<u>12.4.8.</u> ${}_{e}L^{t} \overset{d}{=} \langle {}_{e}P^{t}, {}_{e}M^{t}, \models$ where ${}_{e}P^{t}$ is the set of identities, that is the set of quantifier-free prime formulas containing only one relation symbol $\doteq$ and ${}_{e}M^{t}$ is the class of t-type models satisfying the axioms of equality. /see the proof of Theorem 9.2./

Note, that ${}_{e}P^{t} = \{ (\tau_1 \doteq \tau_2) : \tau_1, \tau_2 \in {}_{1}T^{t} \}$.

a/ Give a complete calculus to ${}_{e}L^{t}$ by the completeness of ${}_{0}L^{t}$.

b/ Give an adequate calculus to ${}_{e}L^{t}$ by exercise 7.

Hint: use the axioms of equality /see proof of Theorem 9.2./.

<u>12.4.9.</u> The Birkhoff inference system:

$\vdash \tau \doteq \tau$ for any $\tau \in {}_{1}T^{t}$

$\tau_1 \doteq \tau_1', \dots, \tau_n \doteq \tau_n' \vdash f(\tau_1, \dots, \tau_n) \doteq f(\tau_1', \dots, \tau_n')$

for any $\tau_1, \tau_1', \dots, \tau_n, \tau_n' \in {}_{1}T^{t}$ and $\langle f, n \rangle \in t''$.

$\tau_1 \doteq \tau_2 \vdash [\tau_1 \doteq \tau_2)[x_i/\tau_3]$ for any $i < \omega$ and $\tau_1, \tau_2, \tau_3 \in {}_{1}T^{t}$.

$\tau_1 \doteq \tau_2, \tau_2 \doteq \tau_3 \vdash \tau_1 \doteq \tau_3$

$\tau_1 \doteq \tau_2 \vdash \tau_2 \doteq \tau_1$ for any $\tau_1, \tau_2, \tau_3 \in {}_{1}T^{t}$.

Show the completeness and adequateness of the Birkhoff inference system w.r.t. ${}_{e}L^{t}$ by Exercise 12.4.8.

Note, that the Birkhoff inference system is just the usual system of equality rewriting rules. This exercise belongs to Proof Theory of first-order logic, because the derivation of the Birkhoff inference system from the calculus of Exercise 12.4.8. is a purely syntactical work.

12.4.1o. Show, that if $\models \exists x_1 \forall y_1 \dots \exists x_n \forall y_n \psi$ then there exist $m < \omega$, $\tau_i^j \in {}_2T^t$ and $f_i \in V_i^F$ for all $i \leq n$, $j \leq m$ such that

$$\models \psi[x_1/\tau_1^1, \dots, x_n/\tau_n^1, y_1/f_1(\tau_1^1), \dots, y_n/f_n(\tau_1^1, \dots, \tau_n^1)] \vee$$
$$\vee \dots \vee \psi[x_1/\tau_1^m, \dots, x_n/\tau_n^m, y_1/f_1(\tau_1^m), \dots, y_n/f_n(\tau_1^m, \dots, \tau_n^m)]$$

/Hint: use Skolem's normal form theorem - Theorem 11.5. -, Theorem 11.1o. and the Compactness Theorem of ${}_0L^t$ - Theorem 5.7. -/.

13. PROOF THEORY OF FIRST-ORDER LOGIC

13.1. Axiomatic calculuses for ${}_1L^t$.

We give an adequate axiomatic calculus for ${}_1L^t$. If the reader is not interested, this calculus can be skipped-we shall not use it later.

Before defining this calculus, we need some auxiliary definitions, from purely technical reasons. Namely, the algorithm ZER changed function variables to function symbols, and we must change them back now.

$t^+ \stackrel{d}{=} \langle t', t'' \cup \{\langle f, n\rangle : f \in V_n^F\}\rangle$

That is, t^+ is an extension of the type t, so that t^+ has enough function symbols and in the same time, the new function symbols are variables viewed from ${}_2L^t$. . So we got rid of the assumption that t has enough function symbols, too.

The algorithm ZER^+ is a version of ZER which uses $ALGTER^{t+}$ instead of $ALGTER^t$ and EXI^+ instead of EXI, where EXI^+ is defined as follows:
from the definition of EXI we know that for any $\varphi \in {}_1F^t$ there is a quantifier-free $\psi \in {}_2F^{+t}$ such that:

$$\varphi \equiv \forall f_1 \dots f_n \exists x_1 \dots x_n \psi .$$

If we consider $\varphi \in {}_1F^t$ as an element of ${}_1F^{t^+}$ the new f'- s appearing in ψ can be chosen from $\bigcup_{n\in\omega} V_n^F$ /see the proof of Corollary 11.7./.
Now, $EXI^+(\varphi) \stackrel{d}{=} \exists x_1 \dots x_n \psi$
Note, that

(i) for any quantifier-free $\rho \in {}_2F^t$ which does not contain variables from $\bigcup_{n\in\omega} V_n^R$:

$${}_2\models^t \rho \quad \text{iff} \quad {}_0\overline{CAL}^{t^+}(\rho) = 1.$$

Recognise, that $\rho \in {}_0F^{t^+}$.

(ii) for any $\varphi \in {}_1S^t$

$${}_1\models^t \varphi \quad \text{iff} \quad {}_2\models^t \bigvee_{i<n} \overline{ZER}^+(i,\varphi) \quad \text{for some } n < \omega.$$

Definition 13.1.

${}_1AXC_0^t$ is an axiomatic calculus defined as follows:

(i) The set of axioms:

$$\{\rho : {}_0\overline{CAL}^{t^+}(\rho) = 1 \; ; \; \rho \in {}_2F^t$$

is quantifier-free $\} \; \bigcup$

$$\bigcup \{ ((\bigvee_{i<n} \overline{ZER}^+(i,\varphi)) \rightarrow \varphi) : \varphi \in {}_1F^t, n < \omega \}$$

(ii) The set of rules of inference is MP i.e.:

$$\{ \langle \{\varphi, (\varphi \rightarrow \psi)\}, \psi \rangle : \varphi, \psi \in {}_1F^t \}$$

This ${}_1AXC_0^t$ is an axiomatic calculus, because the set of axioms is decidable since ${}_0CAL^t$ is a decision calculus and given χ we can "algorithmically" find all the /finitely many/ φ-s and n-s, for which $\chi = \overline{ZER}^+(n,\varphi)$ for any formula containes only finitely many elements from ${}_1T^{t^+}$. /see also Corollary 11.11./ The set MP is also decidable.

Theorem 13.2.

${}_1AXC_0^t$ is an adequate calculus for ${}_1L^t$.

Proof

The proof immediately follows from the completeness theorem of first-order logic /Theorem 12.2./.

▲

Remark:

In Definition 13.1. we extended the complete calculus ${}_1CAL^t$ to an adequate axiomatic one. To get an adequate /but not axiomatic/ calculus it was enough to say that if we want to enumerate the consequences of a formula φ then we have to enumerate all the tautologies of the form $\varphi \to \psi$. However, this is not an axiomatic calculus because $\{\varphi \to \psi : \models (\varphi \to \psi)\}$ is not decidable. We could not use axioms $\bigvee_{i<n} \overline{ZER}(i, \varphi) \to \varphi$ bacause in $_1CAL^t$ we used only the fact that $\models \bigvee_{i<n} \overline{ZER}(i, \varphi)$ implies $\models \varphi$, but $\not\models (\bigvee_{i<n} \overline{ZER}(i, \varphi) \to \varphi)$.

Hence we had to change to ZER^+ where
$\models (\bigvee_{i<n} \overline{ZER}^+(i, \varphi) \to \varphi)$ is already true.

Similar problems arise in general when we want to construct an adequate axiomatic calculus from a simply complete one. In complete calculuses we only have to be careful to get tautologies from tautologies, but in the case of adequate calculuses we want to get from any set of formulas something which is a consequence of that set.

By Theorem 13.2. we have an adequate axiomatic calculus for first-order logic. The search for other /more elegant, more effective etc./ calculuses is a purely syntactical problem and thus belongs to proof theory.

Just as it was the case in the Proof Theory of zero-order logic we want to have an axiomatic calculus defined in an intuitively more natural form.

Now, we define an algorithm which computes the "inverse" of the function PREN. We define this algorithm in the form of an axiomatic calculus.

Definition 13.3. The algorithm PRI is defined as:

1/ the set of axioms $Apri$:

$$Apri \overset{d}{=} \{ \varphi \leftrightarrow \exists x \varphi$$
$$\varphi \leftrightarrow \forall x \varphi$$
$$\varphi \leftrightarrow \neg\neg\varphi$$
$$\neg\neg\varphi \leftrightarrow \varphi$$
$$\varphi \wedge \chi \leftrightarrow \chi \wedge \varphi$$
$$\exists x(\varphi \wedge \psi) \leftrightarrow (\varphi \wedge \exists x \psi)$$
$$\forall x(\varphi \wedge \psi) \leftrightarrow (\varphi \wedge \forall x \psi)$$
$$\exists z \varphi \leftrightarrow \exists x \varphi[z/x] :$$

$\varphi, \chi, \psi \in {}_1F^t$ and x does not occure in φ freely and $z, x \in V$ }

2/ The set of rules of inference:

$$Rpri \overset{d}{=} \{ \langle \{ (\varphi \leftrightarrow \psi), \alpha\varphi\beta \}, \alpha\psi\beta \rangle : \alpha\varphi\beta, \varphi, \psi \in {}_1F^t \}$$

For the notation $\alpha\varphi\beta$ see Exercise 1o.3.4.

Lemma 13.4.

PRI computes the inverse of PREN, more precisely, for any $\varphi \in {}_1F^t$, $PREN(\varphi) \vdash^{PRI} \varphi$ and if $\varphi \vdash^{PRI} \psi$ then also $\varphi \equiv \psi$.

Proof

The proof is immediate by checking the definition of PREN and the fact that all the steps in PREN are included in $Apri$.

Definition 13.5.

The axiomatic calculus ${}_1AXC_1^t$ is defined as follows:

(i) The set of axioms ${}_1Ax_1^t \overset{d}{=} Axi \cup Axii \cup Apri$ where

$$Axi \overset{d}{=} \{ (\varphi[x/\tau] \to \exists x \varphi), \\ (\forall x_1 \dots x_n \exists y \varphi \to \exists f \forall x_1 \dots x_n \varphi[y/f(x_1, \dots, x_n)]) \\ \varphi \in {}_2F^t;\ y, x_1, \dots, x_n \in V,\ f \in V_n^F \}.$$

$Axii$ is the same as ${}_0Ax_2^t$ /of Definition 7.7./ with the only difference that in the case of $Axii$ $\varphi, \psi, \chi \in {}_2F^t$.

(ii) The rules of inference:

${}_1R_1^t \overset{d}{=} Rpri \cup MP \cup RG$ where RG is the set of rules of generalisation, i.e.:

$$RG \overset{d}{=} \{ \langle \varphi, \forall v \varphi \rangle : \varphi \in {}_2F^t \text{ and } v \in \bigcup_{n \in \omega} V_n^F \cup V \}$$

Note, that $Apri$ and $Rpri$ were defined in Definition 13.3.

Before showing the adequateness of ${}_1AXC_1^t$ we state a Lemma. For convenicence, we introduce the following notation.

Notation

$\bigvee_{j \le m} \psi[x_i/\tau_i^j, y_i/f_i(\tau_1^j, \dots, \tau_i^j)]_{i \le n}$ stands for the formula $\psi[x_1/\tau_1^1, \dots, x_n/\tau_n^1, y_1/f_1(\tau_1^1), \dots, y_n/f_n(\tau_1^1 \dots \tau_n^1)] \vee \dots \vee \psi[x_1/\tau_1^m, \dots, x_n/\tau_n^m, y_1/f_1(\tau_1^m), \dots, y_n/f_n(\tau_1^m, \dots, \tau_n^m)]$

Lemma 13.6.

Let $\models \exists x_1 \forall y_1 \dots \exists x_n \forall y_n \psi$. Then there are $m \in \omega$; $\tau_j^i \in {}_2T^t$, $f_i \in V_i^F$ for all $i \le n$; $j \le m$ such that $\models \bigvee_{j \le m} \psi[x_i/\tau_i^j, y_i/f_i(\tau_1^j \dots \tau_i^j)] \dots$

Proof

Let $\models \exists x_1 \forall y_1 \ldots \exists x_n \forall y_n \psi$.

By Skolem's normal form theorem /Theorem 11.5./ this implies that there are $f_i \in V_i^F$ for all $i \leq n$ such that

$\models \forall f_1 \ldots f_n \exists x_1 \ldots x_n \psi[y_i / f_i(x_1, \ldots, x_i)]_{i \leq n}$,

and thus $\exists x_1 \ldots x_n \psi [y_i / f_i(x_1, \ldots, x_i)]_{i \leq n}$.

Using the fact, that

$\psi[y_i / f_i(x_1, \ldots, x_i)]_{i \leq n} = \psi[x_i / \tau_i, y_i / f_i(\tau_1, \ldots, \tau_i)]_{i \leq n}$,

by Theorem 11.1o and the Compactness theorem of zero-order logic /Theorem 5.7./, there are $m \leq \omega$, $\tau_j^i \in {}_2T^t$ for all $i \leq n, j \leq m$ such that

$\models \bigvee_{j \leq m} \psi[x_i / \tau_i^j, y_i / f_i(\tau_1^j, \ldots, \tau_i^j)]_{i \leq n}$

▲

Theorem 13.7.

${}_1AXC_1^t$ is an adequate axiomatic calculus for ${}_1L^t$.

Proof

Because MP is included in the rules of inference it is enough to prove that ${}_1AXC_1^t$ is complete /see Theorem 7.6./.

Let $\varphi \in {}_1F^t$ be a tautology and let

$PREN(\varphi) = \exists x_1 \forall y_1 \ldots \exists x_n \forall y_n \psi$. From this by Lemma 13.6. we have $\models \bigvee_{j \leq m} \psi[x_i / \tau_i^j, y_i / f_i(\tau_1^j, \ldots, \tau_i^j)]_{i \leq n}$.

Since ψ is quantifier-free, by the completeness of ${}_0AX_2^t$

$\vdash \bigvee_{j \leq m} \psi[x_i / \tau_i^j, \ldots]$. /See Lemma 7.8./.

Using axiom $\varphi[x/\tau] \to \exists x \varphi$, Axi and MP ;

$\{ \bigvee_{j \leq m} \psi[x_i / \tau_i^j, \ldots] \} \vdash \exists x_1 \ldots x_n \psi[y_1 / f_1(x_1), \ldots, y_n / f_n(x_1, \ldots, x_n)]$

and using RG $\vdash \forall f_n \exists x_1 \ldots x_n \psi[y_1 / f_1(x_1), \ldots]$

From Axii, Axi and MP

$$\vdash (\forall f_n \exists x_1 .. x_n \varphi[y_1 / f_1(x_1), \dots] \longrightarrow \exists x_1 .. x_n \forall y_n \varphi[y_1/f_1(x_1), .., y_{n-1}/f_{n-1}(x_1, .., x_{n-1})]$$

By this formula and MP

$$\vdash \exists x_1 .. x_n \forall y_n \varphi[y_1/f_1(x_1), .., y_{n-1}/f_{n-1}(x_1, .., x_{n-1})]$$

Now by RG $\vdash \forall f_{n-1} \exists x_1, .. x_n \forall y_n \varphi[y_1/f_1(x_1), \dots]$ etc.

Repeating the last three steps n-times we get

$$\vdash \exists x_1 \forall y_1 \dots \exists x_n \forall y_n \varphi .$$

This formula is $\overline{PREN(\varphi)}$. Now, applying $Apri$ and $Rpri$ by Lemma 13.4. we get $\vdash \varphi$.

▲

<u>Remark:</u>

The calculus ${}_1AXC_1^t$ contains in its axioms and inference rules not only first-order formulas but also second-order ones /which actually do not belong to the language ${}_1L^t$ /. We do not see why would it be necessary to exclude second-order formulas from a first-order calculus. The important thing is that the calculus should be adequate w.r.t. all first-order sentences /and of course sound w.r.t. all formulas it uses/.

This usage of second-order formulas makes the calculus more consize and natural, coincides with the everyday mathematical practice /and so is not at all counter-intuitive/.

<u>Exercises 13.8.</u>

<u>13.8.1.</u> Give a detailed proof of Lemma 13.4.

<u>13.8.2.</u> Construct an axiomatic calculus ${}_1AXC_4^t$ similarly to ${}_1AXC_1^t$ using instead of $Axii$ the original

form of ${}_0Ax_2^{\,t}$ extended to ${}_0Ax_2^{\,t^+}$ and adding to the set of rules of inference the following set:

$$\{\langle\{\bigvee_{i\le n}\varphi_i,\ \varphi_1\to\psi,\dots,\varphi_n\to\psi\},\ \psi\rangle : \varphi_i,\psi\in_1 F^t;\ i\le n,\ n<\omega\}$$

For the definition of t^+ see the first paragraph of Section 13.1.

13.2. The Gödel completeness theorem of first-order logic

Let $\{x_1,\dots,x_n,y_1,\dots y_n\}\subseteq V$ have cardinality $2\cdot n$ i.e. the variables $x_i, y_i\ (i\le n)$ are all distinct.

Let $\models \exists x_1\forall y_1\dots\exists x_n\forall y_n\psi$ be a first-order sentence such that ψ is quantifier-free. Now by Lemma 13.6. there are $m<\omega$; $\tau_i^j\in_2 T^t$; $f_i\in V_i^F$ for all $i\le n; j\le m$ such that

$$\models \bigvee_{j\le m}\psi[x_i/\tau_i^j,\ y_i/f_i(\tau_1^j,\dots,\tau_i^j)]_{i\le n}.$$

To analise this result we define the i-long j-th term row /of course $i\le n, 1\le j\le m$/ as:

$$r_i^j \overset{d}{=} \langle\tau_1^j,\dots,\tau_i^j, f_1(\tau_1^j),\dots,f_i(\tau_1^j,\dots,\tau_i^j)\rangle$$

$R=\{r_i^j : i\le n,\ 1\le j\le m\}$. Note, that r_0^j is the empty row /sequence/.

The binary relation $\lesssim$ is defined on R as:

$$r_i^j\lesssim r_k^l \text{ iff } f_i(\tau_1^j,\dots,\tau_i^j)\in\{\tau_1^l,\dots,\tau_k^l, f_1(\tau_1^l),\dots,f_k(\tau_1^l,\dots,\tau_k^l)\}$$

Note, that $r_i^j\lesssim r_k^l$ iff the last term of r_i^j is a term of r_k^l.

Lemma 13.9.

The binary relation $\precsim$ is antisymmetric.

Proof

1./ if $r_i^j \precsim r_k^\ell$ then $\tau_1^j, \dots, \tau_i^j$ is a subtext of $\tau_1^\ell, \dots, \tau_k^\ell$.

To see this, suppose $r_i^j \precsim r_k^\ell$. Now, by the definition of $\precsim$, either $i \leq k$ and $f_i(\tau_1^j, \dots, \tau_i^j) = f_i(\tau_1^\ell, \dots, \tau_i^\ell)$ or $f_i(\tau_1^j, \dots, \tau_i^j) = \tau_p^\ell$ for some $p \leq k$. In both cases $\tau_1^j, \dots, \tau_i^j$ is a subtext of $\tau_1^\ell, \dots, \tau_k^\ell$. Thus 1.) is proved.

2./ By (1.), $r_i^j \precsim r_k^\ell$ and $r_k^\ell \precsim r_i^j$ implies that $r_i^j = r_k^\ell$.

▲

Corollary 13.1o.

Every subset of the set R has a maximal element w.r.t. $\precsim$.

Proof

The proof follows from antisimmetricity of $\precsim$ and finiteness of R.

▲

Let us have a fixed /but arbitrary/ one-one function

$$z : \{\tau_i^j : i \leq n, j \leq m\} \longrightarrow (V \setminus \{x_1, \dots, x_n, y_1, \dots, y_n\})$$

Now we define /for any $k \leq n$, $1 \leq j \leq m$/:

$$\varphi_k^j \overset{d}{=} \exists x_{k+1} \forall y_{k+1} \dots \exists x_n \forall y_n \, \varphi[x_i / z(\tau_i^j), y_i / z(f_i(\tau_1^j, \dots, \tau_i^j))]_{i \leq k}.$$

Note, that the set of free variables of φ_k^j is

$$\{z(\tau_i^j),\ z(f_i(\tau_1^j, \dots, \tau_i^j)) : 1 \leq i \leq n\}.$$

Lemma 13.11.

$$\models \bigvee_{j \leq m} \varphi_n^j \quad \text{iff} \quad \models \bigvee_{j \leq m} \varphi[x_i / \tau_i^j, y_i / f_i(\tau_1^j, \tau_i^j)]$$

Proof

The proof is immediate by the basic properties of the quantifier-free tautologies /see Corollary 5.12./.

▲

Lemma 13.12.

For any $h:\{1,\dots,m\}\to\{0,\dots,n\}$ there is a $k\le m$ for which

(i) the variable symbol $z(f_{h(k)}(\tau_1^k,\dots,\tau_{h(k)}^k))$ does not occure in $\bigvee_{\substack{j\le m\\ j\ne k}} \varphi^j_{h(j)}$.

(ii) $\bigvee_{j\le m} \varphi^j_{h(j)} \models \Big(\big(\bigvee_{\substack{j\le m\\ j\ne k}} \varphi^j_{h(j)}\big) \vee \forall z(f_{h(k)}(\tau_1^k,\dots,\tau_{h(k)}^k))\varphi^k_{h(k)}\Big)$

Proof

The proof of (i) is immediate from Corollary 13.1o. The set $\{\tau^j_{h(j)} : j\le m\}$ has a maximal element, say $\tau^k_{h(k)}$. Thus $z(f_{h(k)}(\tau_1^k,\dots,\tau_{h(k)}^k))$ does not occure in any $\varphi^j_{h(j)}$ if $j\ne k$

(ii) is immediate from (i) and the fact that if x does not occure in φ then $(\varphi\vee\chi)\models\varphi\vee\forall x\chi$.

▲

Now, we define the most frequently used axiomatic calculus of first-order logic, often called Hilbert-Glivenko calculus. This calculus can be found in every introductory textbook on mathematical logic.

Definition 13.13.

The algorithm ${}_1AXC_2^t$ is defined as follows:

(i) The set of axioms:

${}_1Ax_2^t \overset{d}{=} Axi_2 \cup Axii_2 \cup Apri$ where:

$Axi_2 \overset{d}{=} \{(\varphi\vee\psi)\to\varphi\vee\exists x\psi)$ /a/

$\forall x(\varphi\vee\psi)\to(\varphi\vee\forall x\psi)$ /b/

$\forall z\varphi \leftrightarrow \forall x\varphi[z/x]$ /c/

$\varphi, \psi \in {}_1F^t$ and x does not occure in φ and $x, z \in V \}$.

$Axii$ is the same as ${}_0Ax_2^t$ of Definition 7.7. with the only difference that in the case of $Axii$, $\varphi, \psi, \chi \in {}_1F^t$.

(ii) The rules of inference: ${}_1R_2^t = {}_1R_1^t$
Note, that ${}_1R_1^t$ was defined in Definition 13.5. and $Axpri$ in Definition 13.3.

Note, that the algorithm PRI is included in ${}_1AXC_2^t$ because of $Apri$ and $Rpri$.
Also note, that ${}_1AXC_2^t$ is complete w.r.t. all first-order formulas which have the "form" of a propositional /zero-order/ tautology because of $Axii$ and MP.

Theorem 13.14. /The Gödel completeness theorem/
The calculus ${}_1AXC_2^t$ is adequate for the language $\langle {}_1S^t, M^t, \models \rangle$.

Proof
During the proof we use the following notations. Instead of $\vdash_{{}_1AXC_2^t}$ we write $\vdash$ in the general case and write $\vdash^{(a)}$ if the element (a) of Axi_2 is used only in the derivation.
Because MP, $Apri$ and $Rpri$ are included in ${}_1AXC_2^t$ by Lemma 13.4. it is enough to show that ${}_1AXC_2^t$ is complete w.r.t. the sentences in prenex normal form.

Let ψ be quantifier-free and let $\models \exists x_1 \forall y_1 \ldots \exists x_n \forall y_n \psi$ be a first-order tautology.
Now, by Lemmas 13.6. and 13.11 there are $m \leq \omega$, $\tau_i^d \in {}_2T^t$, $i \in I$.

for all $i \le n, j \le m$ such that $\vdash \bigvee_{j \le m} \psi_n^j$. Since ψ_n^j is a quantifier-free, by the completeness of ${}_0Ax_2^t$ and MP for ${}_0L^t$ /See Lemma 7.8./ we have $\vdash \bigvee_{j \le m} \psi_n^j$. By Lemma 13.12 there is a $k \le m$ such that

$$\vdash^{(b,a)} \Big(\bigvee_{\substack{j \le m \\ j \ne k}} \psi_n^j\Big) \vee \exists x_n \forall y_n \big(\psi_n^k [z(\tau_n^k)/x_n, z(f_n(\tau_1^k, .., \tau_n^k))/y_n]\big)$$

using RG.

Notice that the derived formula is $\Big(\bigvee_{\substack{j \le m \\ j \ne k}} \psi_n^j\Big) \vee \psi_{n-1}^k$

Now we can repeat this step in an induction procedure: by Lemma 13.12 for any $h : \{1, .., m\} \to \{0, .., n\}$ there is a $1 \le k \le m$ for which:

$$\Big(\bigvee_{j \le m} \psi_{h(j)}^j\Big) \vdash^{(a,b)} \Big(\bigvee_{\substack{j \le m \\ j \ne k}} \psi_{h(j)}^j\Big) \vee \exists x_k \forall y_k \psi_{h(k)}^k \Big[z(\tau_{h(k)}^k)/x_{h(k)}, z(f_{h(k)}(\tau_1^k, .., \tau_{h(k)}^k))/y\Big]$$

Notice, that the derived formula is

$$\Big(\bigvee_{\substack{j \le m \\ j \ne k}} \psi_{h(j)}^j\Big) \vee \psi_{h(k)-1}^k .$$

Thus by induction and transitivity of $\vdash^{(a,b)}$ we have

$$\bigvee_{j \le m} \psi_n^j \vdash^{(a,b)} \bigvee_{j \le m} \psi_0^j .$$

Since $\psi_0^j = \exists x_1 \forall y_1 \ldots \exists x_n \forall y_n \psi$ by $Ax\ddot{i}_2$ and MP we get

$\vdash \exists x_1 \forall_1 \ldots \exists x_n \forall y_n \psi$.

Note, that by this theorem we have a new, elementary proof of Gödel's completeness theorem. /The important feature of Gödel's completeness theorem is that it states the completeness of an axiomatic calculus using exclusively first-order formulas, and given in a formal fashion like ${}_1AxC_2^t$ /.

<u>Theorem 13.15.</u> /Herbrand's Theorem/

Let $\varphi \in {}_1F^t$ and let t have enough function symbols. Then:

$\vdash_{{}_1AxC_2^t} \varphi$ iff there is an $n < \omega$ for which

$$\vdash_{{}_0AxC_1^t} \bigvee_{i < n} \overline{ZER}(\varphi, i).$$

Proof

The theorem is immediate from the completeness theorems of ${}_1AXC_2^t$ and ${}_0AXC_1^t$, the compactness theorem of ${}_0L^t$ and the basic properties of ZER see Theorem 11.1o.

▲

Exercises 13.16.

13.16.1. Let ${}_1AXC_3^t$ be an equivalent calculus to ${}_1AXC_2^t$ with the only difference that

${}_1Ax_3^t \overset{d}{=} Axi_3 \cup Axii_2 \cup Apri$, where

$Axi_3 \overset{d}{=} \{ \forall z\varphi \leftrightarrow \forall x\varphi[z/x],\ \varphi \to \exists x\varphi$

$\varphi \in {}_1F^t$, x does not occur in φ and $x, z \in V \}$.

Show that ${}_1AXC_3^t$ is an adequate calculus for the language $\langle {}_1S^t, M^t, \models \rangle$.

/Hint: Use ${}_1Ax_3^t$ and ${}_1R_2^t$ to derive Axi_2 /

13.16.2. Add the following rules of inference

$\{ \langle \{\forall x\varphi\}, \varphi \rangle : \varphi \in {}_1F^t \}$ to ${}_1R_2^t$.

Show that this version of ${}_1AXC_2^t$ is an adequate calculus for ${}_1L^t$.

In this section we investigate only the completeness of the calculuses because any complete calculus can be easily extended to an adequate one as it was shown in Section 2. /see also Definition 6.7. and Theorem 6.8./.

Now we introduce a calculus which is a more effective version of ${}_1CAL_0^t$.

14. SOME MORE EFFECTIVE CALCULUSES FOR ${}_1L^t$

Theorem 14.1.

Let $d = \langle d', d'' \rangle$ be a type and $d' \subseteq t', d'' \subseteq t''$; if $\varphi \in {}_2F^d$ then ${}_2\models^d \varphi$ iff ${}_2\models^t \varphi$.

With other words the theorem states that:

$(\forall \mathfrak{N} \in M^d) \mathfrak{N} \models^d \varphi$ iff $(\forall \mathfrak{B} \in M^t) \mathfrak{B}\,{}_2\models^t \varphi$.

Proof

The proof is an immediate consequence of the definition of $\models$.

▲

Definition 14.2.

Let $\varphi \in {}_1F^t$. We define a type ${}_\varphi t \stackrel{d}{=} \langle t', {}_\varphi t'' \rangle$, where ${}_\varphi t''$ is the smallest subset of t'' for which $\varphi \in {}_1F^{\langle t', {}_\varphi t'' \rangle}$ and ${}_\varphi t''(c_o) = 0$./Remember, that by definition, any type contains c_o as a constant symbol./ With other words ${}_\varphi t''$ is the restriction of t'' to c_o and the function symbols occurring in φ.

Note, that ${}_\varphi t''$ is always finite.

Note, that by Theorem 14.1.

$$\models \varphi \qquad \text{iff} \qquad {}_1\models^{\varphi t} \varphi \; .$$

Definition 14.3.

To any $\varphi \in {}_1F^t$ we define

$${}_\varphi ATL \stackrel{d}{=} ALGTER_0^{\varphi t}$$ /see Section 4.1./.

Now we define a modified version of the algorithm

$$\overline{ZER} : {}_1S^t \times \omega \rightarrow {}_0F^t \; .$$

Definition 14.4.

We define the algorithm $\overline{ZER}_1 : {}_1S^t \times \omega \rightarrow {}_0F^t$ as follows for any $\varphi \in {}_1S^t$ and $n < \omega$:

$\overline{ZER}_1(\varphi, n) = \psi[X / \overline{ALGTER}^{\varphi t}(n)]$, where $\exists x \psi = \overline{EXI}(\varphi)$ and ψ is quantifier-free. The definition to the case $\exists x_1 \dots x_n \varphi$ is similar. /see the definition of ZER which can be found in the proof of Theorem 11.1o./.

Lemma 14.5.

$$\models \varphi \quad \text{iff} \quad \models \bigvee_{i<\omega} \overline{ZER_1}(\varphi, i).$$

Proof

The proof is immediate from Theorem 11.1o.

Definition 14.6.

(i) The algorithm $_\varphi\overline{ZER_1} : \omega \longrightarrow {}_0F^t$ is defined as $_\varphi\overline{ZER_1}(n) \stackrel{d}{=} \overline{ZER_1}(\varphi, n)$ for any $\varphi \in {}_1S^t$ and $n < \omega$.

(ii) The algorithm $_1\overline{CAL}^t_5 : {}_1S^t \longrightarrow \{0, 1\}$ is defined as $_1\overline{CAL}^t_5(\varphi) \stackrel{d}{=} {}_0\overline{GCAL}^t(_\varphi\overline{ZER_1})$ for any $\varphi \in {}_1S^t$.

Theorem 14.7.

$_1CAL^t_5$ is a complete calculus for $\langle {}_1S^t, M^t, \models \rangle$.

Proof

The proof is analogious to the completeness proof of $_1CAL^t$. /Theorem 12.2./

Remark:

The calculus $_1CAL^t_5$ is more effective then the calculus $_1CAL^t$ because when proving $\models \varphi$ $\overline{ZER_1}$ uses only the elements of $_0T^{\varphi t}$.

Now we introduce the so called "Resolution" calculuses. To do this we need modified versions of the algorithms

${}_{0}GCAL^{t}$ of Definition 6.1o. and ${}_{0}AGCAL^{t}$ of Definition 6.12.

Definition 14.8.

The algorithm ${}_{0}GCAL_{1}^{t}$ is defined on the set of algorithms as follows:

for any given algorithm ALG

${}_{0}\overline{GCAL_{1}^{t}}(ALG)=1$ iff

max $\{{}_{0}\overline{RCAL}^{t}(\bigvee_{i<n}\overline{ALG}(i)) : n<\omega\}=1$ and it is undefined otherwise.

The definition of ${}_{0}RCAL^{t}$ can be found in Example 7.1.

Theorem 14.9.

For any algorithm $ALG : \omega \to {}_{0}\underline{F}^{t}$

$\models \bigvee_{i<\omega} ALG(i)$ iff $\models {}_{0}GCAL_{1}^{t}(ALG)=1$

Proof

The proof is analogious to the proof of Theorem 6.11.

▲

We introduce two kinds of Resolution calculuses: Direct-Resolution and Indirect /refutation/ Resolution.

Definition 14.1o.

The Direct-Resolution calculus ${}_{1}DRCAL^{t}$ is defined as follows:

${}_{1}\overline{DRCAL}^{t} : {}_{1}S^{t} \to \{0,1\}$ such that:

for any $\varphi \in {}_{1}S^{t}$ ${}_{1}\overline{DRCAL}^{t}(\varphi) \stackrel{d}{=} {}_{0}\overline{GCAL_{1}}^{t}(\varphi^{ZER})$

/for φ^{ZER} see Definition 14.6./

●

Theorem 14.11.

The Direct-Resolution calculus ${}_1DRCAL^t$ is complete for $\langle {}_1S^t, M^t, \models \rangle$.

Proof

The proof is entirely analogous to the proof of the completeness theorem of ${}_1CA'^t$ /Theorem 12.2./.

▲

Remark:

The Direct-Resolution algorithm proves a sentence $\varphi \in {}_1S^t$ in the following way.

First it converts φ into existential form $\overline{EXI(\varphi)}$. Then translates $\overline{EXI(\varphi)}$ into a zero-order sentence: $\overline{ZER_1(\varphi, 1)}$. Then decides by the zero-order calculus ${}_0RCAL^t$ if $\overline{ZER_1(\varphi, 1)}$ is a tautology or not. If not, then takes $(\overline{ZER_1(\varphi, 1)} \vee \overline{ZER_1(\varphi, 2)})$, decides this by ${}_0RCAL^t$ etc. Summing up: Direct-Resolution takes the existential form of φ and tries to prove a zero-order tautology which implies φ.

Indirect-Resolution proves a $\varphi \in {}_1S^t$ in the following way. It takes the negation of φ $(\neg\varphi)$, takes the universal form of $\neg\varphi$ and tries to prove the negation of a zero-order consequence of $\neg\varphi$.

Note, that the universal form of $\neg\varphi$ is equivalent to $\neg\overline{EXI(\varphi)}$. Before defining Indirect-Resolution we have to define the algorithms UNI, ZER etc.

Definition 14.12.

1./ Let $\varphi \in {}_1S^t$ and $\overline{EXI}(\varphi) = \exists x_1 .. x_n \psi$ where ψ is quantifier-free. Then

$$\overline{UNI}(\varphi) \overset{d}{=} \forall x_1 \ldots x_n \neg \psi$$

2./ Let $\varphi \in {}_1S^t$ and $\overline{UNI}(\varphi) = \forall x \chi$, where χ is quantifier-free. Then $_{\varphi}ZER\_(n) \overset{d}{=} \chi[x/\overline{ALGTER}^{\chi^t}(n)]$. The definition for $\forall x_1 .. x_n \chi$ is analogous.

3./ The algorithm $RCAL\_$ is entirely analogous to $_0RCAL^t$, the only difference is that $RCAL\_$ starts from the conjunctive normal for of the zero-order sentence to be refuted. See Example 7.1. For example let this conjunctive normal form be:

$$(a \vee b \vee \neg c) \wedge (a \vee \neg b \vee \neg c) \wedge (\neg a \vee \neg c) \wedge c.$$

from this, form the matrix:

1	a	b	$\neg c$
2	a	$\neg b$	$\neg c$
3	$\neg a$		$\neg c$
4			c

Resolve the rows of the matrix until the empty row is found e.g. resolving rows 1 and 2 we get the row.

5	a		$\neg c$

If the empty row is reached then the result of $RCAL\_$ is 0.

Definition 14.13.

The Inverse-Resolution calculus $IRCAL$ is defined as follows:

$\overline{IRCAL} : {}_1S^t \to \{0,1\}$ such that, for any $\varphi \in {}_1S^t$:

$$\overline{IRCAL}(\varphi) \stackrel{d}{=} 1 - (\min\{\overline{RCAL}_{-}(\textstyle\bigwedge_{i<n} \neg\varphi ZER(i)) : n < \omega\})$$

Theorem 14.14.

The calculus IRCAL is complete for $\langle {}_1S^t, M^t, \models \rangle$.

Proof

The proof is entirely analogous to the proof of the completeness theorem of DRCAL /see Theorem 14.11./.

$\blacktriangle$

Remark:

The most frequently used calculus in automatic theorem-proving is Inverse-Resolution and this is sometimes simply called Resolution.

15. UNDECIDABILITY OF NON-ZERO-ORDER LOGICS

Now we have to go through on some aspects of the concept of algorithm more precisely than in Section 2.

Definition 15.1.

Let Z be a finite set of symbols. Then, Z^* is the set of all finite strings of symbols from Z. E.g. if $Z = \{a, b\}$ then $a, aa, ab, baa \in Z^*$ etc.

Notation: $\mathcal{Al}$ denotes the set of algorithms.

We postulate the following about the concept of algorithm:

<u>Postulate 15.2.</u>

(i) There existst a finite set of symbols Z such that $\mathcal{Al} \subseteq Z^*$.

(ii) Each algorithm $ALG \in \mathcal{Al}$ defines a partial function $\overline{ALG}$ from Z^* into Z^*.

There exists an algorithm $DIAG \in \mathcal{Al}$ and a fixed element K_o of Z^* such that, for all $ALG \in Z^*$

$$\overline{DIAG}(ALG) = \begin{cases} ALG & \text{if } \overline{ALG}(ALG) = K_o \\ \text{undefined otherwise.} \end{cases}$$

Postulate (i) states that there is a finite set of symbols Z by which each algorithm can be described.

Postulate (ii) states that each algorithm $ALG \in \mathcal{Al}$ has a meaning: the execution of ALG. From this meaning we consider only the partial function ALG neglecting all the other aspects of ALG's meaning.

About postulate (iii) we only note, that for any $A, B, ALG \in Z^*$ the statement $\overline{ALG}(A) = B$ means that $ALG \in \mathcal{Al}$ and if we execute ALG for the input A then the algorithm ALG stops in a finite number of steps and gives B as result /output/.

<u>Remark:</u>

In general the partial function ALG is not total on Z^* that is $Do\,\overline{ALG} \subseteq Z^*$ but $Do\,\overline{ALG} \neq Z^*$. $A \in Do\,\overline{ALG}$ iff the execution of ALG for A as input stops in a finite number of steps and the result is B for some $B \in Rg\,\overline{ALG}$.

Notation:

$K_0 \neq K_1$ are two distinguished fixed elements of Z^*.

Remark:

Sometimes we define algorithms as functions with more than one arguments. E.g. $\overline{ALG}$ might be a partial function on $Z^* \times Z^*$. We do not carry through precisely this generalisation of the concept algorithm because it is a trivial matter e.g. it can be done by including a special separating symbol into Z and stating some restriction on $\mathcal{Al}$.

Definition 15.3.

A set $H \subseteq Z^*$ is enumerable iff there is an algorithm $EN \in \mathcal{Al}$ such that $Rg\ \overline{EN} = H$.

Definition 15.4.

A set $H \subseteq Z^*$ is decidable iff there is an algorithm $DE \in \mathcal{Al}$ such that

(i) $Do\ \overline{DE} = Z^*$ and $Rg\ \overline{DE} = \{K_0, K_1\}$

(ii) for any $X \in Z^*$: $\overline{DE}(X) = K_1$ iff $X \in H$.

Now we turn to a basic property of algorithms which is strongly related to the negative result called Church's Theorem.

Theorem 15.5.

For any concept of algorithm satisfying Postulate 15.2., there exists a set $H \subseteq Z^*$ which is enumerable but not decidable.

Proof

We show that $Rg\ \overline{DIAG}$ is such a set. Of course it is enumerable. We show that it is undecidable.

Suppose that $Rg\ \overline{DIAG}$ is decidable. Then there exists an algorithm AL for which:

(i) $Do\ \overline{AL} = Z^*$ and $Rg\ \overline{AL} = \{K_0, K_1\}$

(ii) for all $X \in Z^*$

$\overline{AL}(X) = K_1$ iff $X \in Rg\ \overline{DIAG}$.

Investigate $\overline{AL}(AL)$. By (i) it is either K_0 or K_1.

Suppose $\overline{AL}(AL) = K_1$. In this case by (ii) $AL \in Rg\ \overline{DIAG}$.

That is there is an Y such that $\overline{DIAG}(Y) = AL$. By the definition of DIAG /see Postulate 15.2.(iii)/ $AL = Y$ and $\overline{AL}(AL) = K_0$ which is contradiction.

Suppose now $\overline{AL}(AL) = K_0$. Thus by the definition of DIAG, $\overline{DIAG}(AL) = AL$. Hence $AL \in Rg\ \overline{DIAG}$ which is a contradiction by (ii).

Since (i) implies there is not third case and so $Rg\ \overline{DIAG}$ is undecidable.

▲

Postulate 15.6.

There exists an algorithm WAT such that:

$Rg\ \overline{WAT} = \{K_0, K_1\}$ and for any $A, B, C \in \mathcal{Al}$ and $Do\ \overline{C} = Z^*$ and $X \in Z^*$

the algorithm $\overline{WAT(A,B,C)} \in \mathcal{Al}$ satisfies:

(i) if $\overline{\overline{WAT(A,B,C)}}(X) = K_1$ then $\overline{A}(X) \in Rg\ \overline{C}$

(ii) if $\overline{\overline{WAT(A,B,C)}}(X) = K_0$ then $\overline{B}(X) \in Rg\ \overline{C}$

(iii) if $\overline{\overline{WAT(A,B,C)}}(X)$ is undefined then $\overline{A}(X), \overline{B}(X) \notin Rg\ \overline{C}$.

This postulate sais that if we are given three algorithms it can be "wathched" whether the value of one of the two first algorithms /an X/ occures in the range of the third.

Definition 15.7.

Let $L = \langle F, M, \models \rangle$ be a language. We say that set $B \subseteq Z^*$ is definable in the model $\mathfrak{A} \in M$ iff there are $Y_B, N_B \in \mathfrak{A}$ such that, for all $X \in Z^*$:

(i) $X \in B$ iff $\mathfrak{A} \models \overline{Y_B}(X)$

(ii) $X \notin B$ iff $\mathfrak{A} \models \overline{N_B}(X)$

Definition 15.8.

The language $\langle F, M, \models \rangle$ has expressive power enough iff there is a model $\mathfrak{A} \in M$ such that all enumerable subsets of Z^* are definable in $\mathfrak{A}$.

Definition 15.9.

(i) The formula $\varphi \in F$ describes a model $\mathfrak{A} \in M$ iff for all $\psi \in F$ $\varphi \models \psi$ iff $\mathfrak{A} \models \psi$.

(ii) The set of formulas $\Sigma \subseteq F$ describes $\mathfrak{A} \in M$ iff for all $\psi \in F$ $\Sigma \models \psi$ iff $\mathfrak{A} \models \psi$.

Theorem 15.1o. /Undecidability theorem/

For any concept of algorithm satisfying postulates 15.2. and 15.6. the following holds:

(i) For any adequate logic if its language $\langle F, M, \models\rangle$ has expressive power enough then there is a model $\mathfrak{A} \in M$ which cannot be described by a single formula $\varphi \in F$.

(ii) Moreover, if the logic is adequate in the generalised sense /see Theorem 6.13/ then there is a model which cannot be described by any enumerable set of formulas.

<u>Proof</u>

Let $L = \langle F, M, \models\rangle$ have expressive power enough. Let $\langle L, \dot{C}\dot{A}\dot{L}\rangle$ be an adequate logic. Let $\mathfrak{A} \in M$ be the model which gives the property "expressive power enough" /see Definition 15.8./.

Suppose that φ describes $\mathfrak{A}$. From this hipothesis we show that every enumerable subset of Z^* is also decidable. Which contradicts Theorem 15.5.

Let $B \subseteq Z^*$ be enumerable. Thus B is definable in $\mathfrak{A}$ /see Definition 15.8./ and so there are $Y_B, N_B \in \mathcal{A}\ell$ /see Definitions 15.7./. Since CAL is adequate, $\overline{CAL}(\varphi)$ is an algorithm which enumerates the set of all consequences of φ i.e. $Rg\,\overline{CAL\,\varphi} = \{\psi \in F : \mathfrak{A} \models \psi\}$.

Now, we prove that the algorithm $\overline{WAT}(Y_B, N_B, \overline{CAL}(\varphi))$ decides the set B.

Let $X \in Z^*$ be arbitrary.

By Definition 15.7. we have $X \in B$ iff $\mathfrak{A} \models \overline{Y}_B(X)$

But by adequateness of CAL:

$\mathfrak{A} \models Y_B(X)$ iff $\overline{Y}_B(X) \in Rg\,\overline{\overline{CAL}(\varphi)}$

similarly we get $X \notin B$ iff $\overline{N}_B(X) \in Rg\,\overline{CAL}(\varphi)$.

Since either $x \in B$ or $x \notin B$ holds, the algorithm $\overline{WAT(Y_B, N_B, \overline{CAL}(\varphi))}$ is everywhere defined and decides B /because
$\overline{WAT(Y_B, N_B, \overline{CAL}(\varphi))}(X) = K_1$ iff $X \in B$./

This contradicts Theorem 15.5. and thus there exists no $\varphi \in F$ which would describe $\mathfrak{A}$. The proof of (ii) is similar.
▲

Remark:

The above theorem is also called incompleteness theorem because it states that there exists no complete calculus for the language $\langle F, \{\mathfrak{A}\}, \models \rangle$ see e.g. [3].

Now we return to the languages ${}_1L^t$, ${}_2L^t$.

Definition 15.11.

Let $t'(r) = 3$ and for all $X \in Z^*$ $t''(X) = 0$

We define $\mathfrak{L} \in M^t$ as follows:

$\mathfrak{L}_0 = C = Z^*$

$\mathfrak{L}_r \overset{d}{=} \{\langle ALG, X, Y\rangle : ALG \in \mathcal{Al}, \overline{ALG}(X) = Y\}$

and for any $X \in Z^*$

$\mathfrak{L}_X \overset{d}{=} X$.

Theorem 15.12. /Undecidability theorem of first-order logic/

In the language ${}_1L^t$ the model $\mathfrak{L}$ is not describable by any enumerable set of formulas.

Proof

By the proof of Theorem 15.1o. it is enough to show that in $\mathfrak{L}$ every enumerable set is definable, since ${}_1L^t$ can be extended to an adequate logic in the generalised sense.

Let ENU enumerate the set B in $\mathcal{L}$. Now

$$X \in B \quad \text{iff} \quad \mathcal{L} \models \exists v\, \tau(ENU, v, X)$$

$$X \notin B \quad \text{iff} \quad \mathcal{L} \models \neg\exists v\, \tau(ENU, v, X)$$

▲

Definition 15.13.

Let $t'(<) = 2$, $t''(\cdot) = t''(+) = 2$, $t''(\theta) = t''(1) = 0$, $t''(s) = 1$.

the model $\mathfrak{N} \in M^t$ is defined as:

$\mathfrak{N}_0 = N \stackrel{d}{=} \omega$

$\mathfrak{N}_+$ is the usual addition on the natural numbers.

Similarly $\mathfrak{N}.$.

$\mathfrak{N}_\theta \stackrel{d}{=} 0$

$\mathfrak{N}_1 \stackrel{d}{=} 1$

$\mathfrak{N}_s(n) = n+1$ for any $n \in \omega$

$\mathfrak{N}_< = \{\langle n, m\rangle : n < m\}$.

Corollary 15.14. /Gödel's incompleteness theorem/

In the language ${}_1L^t$ the model $\mathfrak{N}$ is not describable by any enumerable set of formulas.

Proof

We omit the proof here because it uses the idea of chosing $\mathfrak{N}$ to be the set of recursive functions /on ω /. See e.g. [3]

▲

Theorem 15.15.

There is no complete calculus for ${}_2L^t$.

Proof

Here we only give the idea of the proof.

Choose a concrete definition of $\mathcal{N}$ as simple as possible, which satisfies Postulates 15.2 and 15.6. By this the model $\mathcal{L}'$ is fixed. Find a second-order formula $\varphi \in {}_2F^t$ which describes $\mathcal{L}'$.

Now by Theorem 15.1o and 15.12 there can not be a complete calculus for ${}_2L^t$.

For example take $\mathcal{N}$ as defined in definition 15.13. By Corollary 15.14 $\mathcal{N}$ can not be described in any complete logic containing ${}_1L^t$. Since ${}_2L^t$ containes ${}_1L^t$ if we find a formula in ${}_2L^t$ which describes $\mathcal{N}$ then there is no complete calculus for ${}_2L^t$. The following finite set of formulas describes $\mathcal{N}$:

Let $P \in V_1^R$

$$\forall P[(P(\theta) \wedge \forall x\,(P(x) \rightarrow P(s(x))) \rightarrow \forall x P(x)]$$

$$\forall x\,(x \not< \theta)$$

$$\forall x \forall y\,(x < y \rightarrow (x \neq y \wedge y \not< x))$$

$$\forall x y\,(x < y \vee y < x)$$

$$\forall x y z\,(x < y \wedge y < z \rightarrow x < z)$$

$$\forall x\,(x < s(x))$$

$$\forall x y\,(x < y \rightarrow y \not< s(x))$$

φ is a formula which defines $+$, $\cdot$ and 1 in terms of $s, \theta, <$.

The conjunction of the above formulas describes $\mathcal{N}$.

▲

16. SOME MODEL THEORETICAL INVESTIGATIONS

In this work model theory was not investigated into a great detail. Now we introduce one of the most important tools of model theory: reduced products. More can be found about the model-theory of classical logic in [4]. In this section we shall use reduced products to prove the compactness theorem and the generalised adequateness theorem for ${}_1L^t$. To do this we shall not use the results proved so far to illustrate the powerfulness of reduced products.

The concept of submodel serves to analyse one model. Now we introduce a tool to compose more models into one. For sake of flexibility we choose "reduced product". $Sb\, I$ stand for the set of all subsets of I.

Let I be a set, and $D \subseteq Sb\, I$.

Let $(\forall i \in I)\ \mathfrak{A}^i \in M^t$.

__Definition 16.1.__

The __D-reduced product__ of the system $\langle \mathfrak{A}^i \rangle_{i \in I}$ is denoted by $\mathbb{P}_D \langle \mathfrak{A}^i \rangle_{i \in I}$ or sometimes we write less precisely $\mathbb{P}_D \mathfrak{A}^i$ and it is defined as follows:

$$(\mathbb{P}_D \mathfrak{A}^i)_0 \overset{d}{=} \mathop{\mathbb{P}}_{i \in I} A^i \overset{d}{=} \{ s \in {}^I\bigcup_{i \in I} A^i : (\forall i \in I)\, s_i \in A^i \}.$$

$$(\forall t''(f)=n)\ (\mathbb{P}_D \mathfrak{A}^i)_f (s^1, \dots, s^n) = \langle \mathfrak{A}^i_f (s^1_i, \dots, s^n_i) \rangle_{i \in I}$$

$$(\forall t'(r)=n)\ \langle s^1, \dots, s^n \rangle \in (\mathbb{P}_D \mathfrak{A}^i)_r \quad \text{iff}$$

$$(\exists H \in D)(\forall i \in H) \langle s^1_i, \dots, s^n_i \rangle \in \mathfrak{A}^i_r .$$

Example 16.2.

We choose the type $t \stackrel{d}{=} \langle \{\langle r,2\rangle\}, \{\langle f,1\rangle\} \rangle$ and

$\mathfrak{A}^0$: $A^0 \stackrel{d}{=} \{a,b\}$, $\mathfrak{A}^0_f(a) \stackrel{d}{=} \mathfrak{A}^0_f(b) \stackrel{d}{=} b$; $\mathfrak{A}^0_r \stackrel{d}{=} \{\langle a,b\rangle\}$.

$\mathfrak{A}^1$: $A^1 \stackrel{d}{=} \{c,d\}$, $\mathfrak{A}^1_f(c) \stackrel{d}{=} d$, $\mathfrak{A}^1_f(d) = c$, $\mathfrak{A}^1_r \stackrel{d}{=} \{\langle c,c\rangle\}$

We illustrate $\mathfrak{A}^0$ and $\mathfrak{A}^1$ on fig. 3.

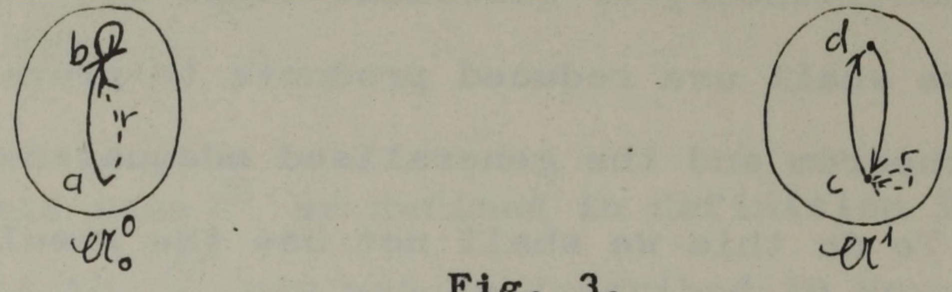

Fig. 3.

From now on we suppose that D is such that if $Y \supseteq X \in D$ then $Y \in D$.

Thus $(\exists H \in D)(\forall i \in H)\, \mathfrak{A}^i \models \varphi$ iff $\{i : \mathfrak{A}^i \models \varphi\} \in D$.

Notation

$$H_\varphi \stackrel{d}{=} \{ i \in I : \mathfrak{A}^i \models \varphi \}$$

We defined D-reduced product such that exactly those prime formulas φ are valid in it, for which $H_\varphi \in D$.

Thus we could reformulate the definition saying simply that:

$\mathbb{P}_D \mathfrak{A}^i \models \varphi$ iff $H_\varphi \in D$;

for all $\varphi \in P^t$.

Lemma 16.3.

$$\overline{(\mathbb{P}_D \mathfrak{A}^i)_r} = \overline{\langle \mathfrak{A}^i_r \rangle_{i \in I}}$$

Proof

The proof follows from the definitions.

▲

Now we try to find such D-s, for which

$$(*) \quad \mathbb{P}_D \mathfrak{A}^i \models \varphi \text{ iff } H_\varphi \in D,$$

for all $\varphi \in {}_1F^t$.

We investigate the requirements in two steps.

a./ Let $H_\psi, H_\chi \in D$.

So $\mathbb{P}_D \mathfrak{A}^i \models \psi, \chi$, thus $\mathbb{P}_D \mathfrak{A}^i \models \psi \wedge \chi$.

But $H_{\psi \wedge \chi} = H_\psi \cap H_\chi$ and thus we need $H_\psi \cap H_\chi \in D$.

For any two sets $\{X, Y \in SbI\}$ we can construct $\langle \mathfrak{A}^i \rangle_{i \in I}$ and φ, ψ such that $X = H_\varphi$, $Y = H_\psi$.

That is, to fulfill $(*)$ D has to be closed to intersection.

b./ 1. Let $H_\psi \in D$.

So $\mathbb{P}_D \mathfrak{A}^i \models \psi$, thus $\mathbb{P}_D \mathfrak{A}^i \not\models \neg\psi$.

But $H_{\neg\psi} = \overline{H_\psi}$ and so to fulfill $(*)$ we need that $\overline{H_\psi} \notin D$.

2. By the properties of ${}_1L^t$ either $\mathbb{P}_D \mathfrak{A}^i \models \varphi$ or $\mathbb{P}_D \mathfrak{A}^i \models \neg\varphi$ and thus to fulfill $(*)$ we need $H_\varphi \in D$ or $\overline{H_\varphi} \in D$.

Since for any $X \subseteq I$ there exists $\langle \mathfrak{A}^i \rangle_{i \in I}$ and φ such that $X = H_\varphi$ we have:

$$X \in D \text{ iff } \overline{X} \notin D.$$

Definition 16.4.

Summing up the requirements, D is such that

0./ $Y \supseteq X \in D \Rightarrow Y \in D$

1./ $X, Y \in D \Rightarrow X \cap Y \in D$

2./ $X \in D \iff \overline{X} \notin D$

Such D-s are called <u>ultrafilters.</u>

<u>Theorem 16.5.</u> /Zorn's Lemma/

Let H be a nonempty set of sets which is closed to the union of nonempty chains. That is if the system $\langle R_j \rangle_{j \in J}$ is such that for all $i, j \in J$ $R_j \in H$ and if $i \leq j$ then $R_i \subseteq R_j$ then also $\bigcup_{j \in J} R_j \in H$. Then H has a maximal element i.e. there is an $X \in H$ for which if $X \subseteq Y \in H$ then $X = Y$.

<u>Proof</u>

Zorn's Lemma is an axiom of set theory so we do not prove it here /see e.g. [4] /.

▲

<u>Lemma 16.6.</u>

For any $R \subseteq SbI$, if R is closed under $\cap$ and $\phi \notin R$ then there exists an ultrafilter D containing R.

<u>Proof</u>

Let $\langle R_j \rangle_{j \in J}$ be a chain in Sb I.

If for all $j \in J$ R_j is closed under $\cap$, $\phi \notin R_j$ and $R \subseteq$

then $\bigcup_{j \in J} R_j$ is also such.

Thus, by Zorn's Lemma, there exists a maximal such set.

This maximal set is an ultrafilter and contains R, by definition 16.4.

▲

<u>Notation:</u>

(i) The function $\mathcal{E}_i : \underset{i \in I}{\mathbb{P}} A^i \longrightarrow \underset{i \in I}{\cup} A^i$ is defined as:

for any $s \in \underset{i \in I}{\mathbb{P}} A^i$: $\mathcal{E}_i(s) = s(i)$.

(ii) The composition $g \circ f$ of two functions g and f is defined as:

for all $x \in Do\, g$:

$g \circ f(x) \overset{d}{=} f(g(x))$.

<u>Theorem 16.7.</u>

If D is an ultrafilter, then

$$\mathbb{P}_D \mathfrak{A}^i \models \varphi \text{ iff } H_\varphi \in D$$

for all $\varphi \in {}_1F^t$.

<u>Proof</u>

We prove that for any function $k : V \longrightarrow \underset{i \in I}{\mathbb{P}} A^i$

$\mathbb{P}_D \mathfrak{A}^i \models \varphi[k]$ iff $H_{\varphi[k]} \in D$, where

$H_{\varphi[k]} \overset{d}{=} \{ i : \mathfrak{A}^i \models \varphi[k \circ \mathcal{E}_i] \}$

The proof goes by induction on the length of φ

1./ The statement is trivial if $\varphi \in {}_1P^t$ by the definition of $\mathbb{P}_D$ /See Definition 16.1./.

2./ Let the statement be true for φ and ψ.

$\mathbb{P}_D \mathfrak{A}^i \models (\varphi \wedge \psi)[k]$ iff

$\mathbb{P}_D \mathfrak{A}^i \models \varphi[k]$ and $\mathbb{P}_D \mathfrak{A}^i \models \psi[k]$ iff /by the induction hypothesis/

$H_{\varphi[k]} \in D$ and $H_{\psi[k]} \in D$ iff

$H_{(\varphi\wedge\psi)[k]} = H_{\varphi[k]} \cap H_{\psi[k]} \in D$ /by the definition of ultrafilters-see Definition 16.4./.

3./ $\mathbb{P}_D \mathfrak{A}^i \models \neg\varphi[k]$ iff
$\mathbb{P}_D \mathfrak{A}^i \not\models \varphi[k]$ iff
$H_{\varphi[k]} \notin D$ iff $H_{\neg\varphi[k]} = \overline{H_{\varphi[k]}} \in D$

4./ Let the statement be true for φ.
$\mathbb{P}_D \mathfrak{A}^i \models \exists x \varphi[k]$ iff
$(\exists a \in \mathbb{P}_{i\in I} A^i) \mathbb{P}_D \mathfrak{A}^i \models \varphi[k(x/a)]$ iff
$(\exists a)\ H_{\varphi[k(x/a)]} \in D$ iff
$H_{\exists x\varphi[k]} \in D$, since by the definition of $H_{\varphi[k]}$ to any φ and k there exists an $a \in \mathbb{P}_D A^i$ such that $H_{\exists x\varphi[k]} = H_{\varphi[k\cdot(x/a)]}$.

<u>Corollary 16.8.</u> /Compactness theorem of first-order language/
Let $\{\varphi_i\}_{i<\omega} \subseteq {}_1S^t{}_)$ then:
$\models \bigvee_{i<\omega} \varphi_i$ iff $\models \bigvee_{i<n} \varphi_i$ for some $n \in \omega$.

That is, for any $\Sigma \subseteq {}_1S^t{}_)$ if to any $\mathfrak{A} \in M^t{}_)$ there is a $\varphi \in \Sigma$ such that $\mathfrak{A} \models \varphi$, then there exists a finite subset Γ of Σ such that to any $\mathfrak{A}$ there is a $\varphi \in \Gamma$ for which $\mathfrak{A} \models \varphi$.

<u>Proof</u>

It is enough to prove that

if $\models \bigvee_{i<\omega} \varphi_i$ then $\models \bigvee_{i<n} \varphi_i$, for some $n<\omega$.

Let us suppose that for any $n\in\omega$, $\not\models \bigvee_{i<n} \varphi_i$, i.e. there is an $\mathfrak{A}^n \in M^t$ for which $\mathfrak{A}^n \not\models \bigvee_{i\leq n} \varphi_i$ and thus:

$\mathfrak{A}^n \models \neg\varphi_i$ for all $i<n$.

Now we construct an ultrafilter D on ω, for which

$\prod_D \mathfrak{A}^n \models \neg\varphi_i$ for all $i\in\omega$. This, of course, will complete the proof.

$R \overset{d}{=} \{\{i : n \leq i < \omega\} : n\in\omega\}$.

Because R is closed w.r.t. $\cap$ by Lemma 16..6. there exists an ultrafilter $D \supseteq R$.

Now $H_{\neg\varphi_i} \supseteq \{j : i \leq j \in \omega\} \in D$ for all $i\in\omega$.

▲

On fig.4. and fig.5. we illustrate D-reduced products of $\mathfrak{A}^0$ and $\mathfrak{A}^1$ for different D-s.

The relation /r/ The function /f/

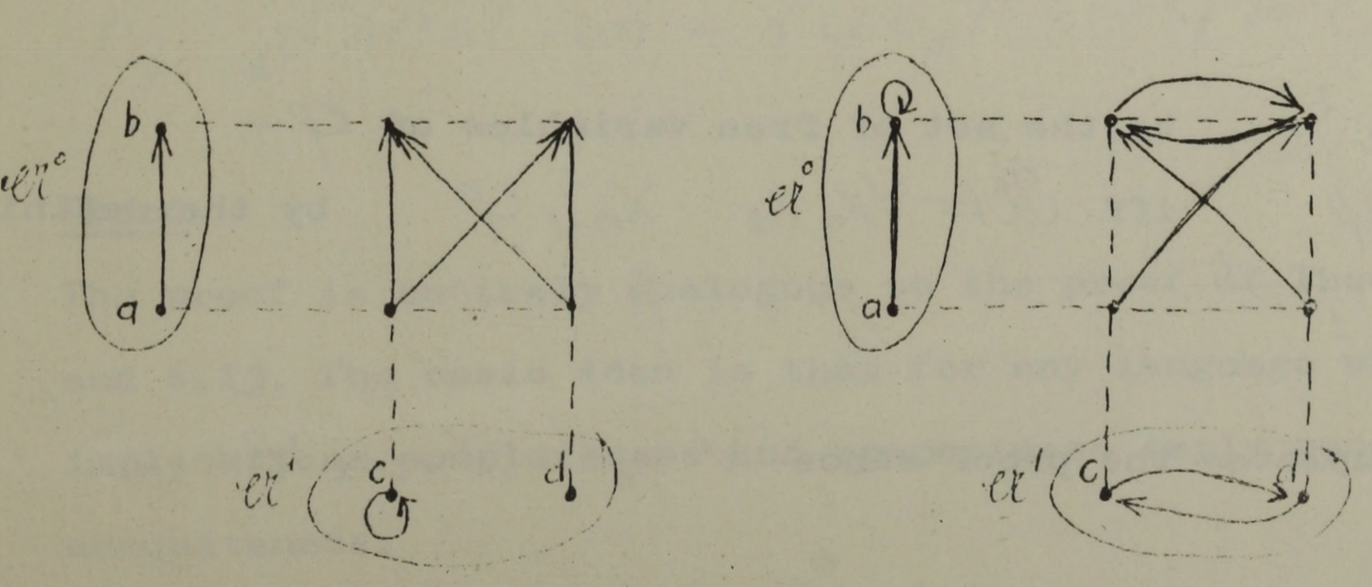

Fig. 4.

$\{\{0\},\{0,1\}\}$ – reduced product

The relation /r/ The function /f/

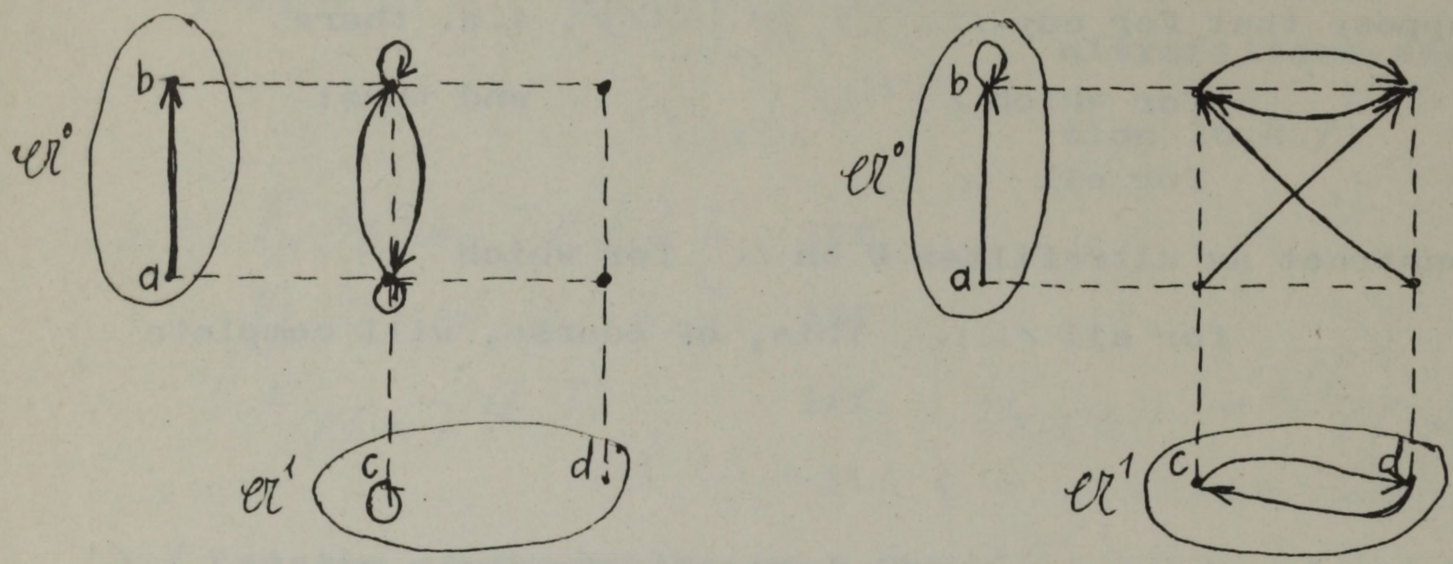

Fig. 5.

$\{\{1\},\{0,1\}\}$ – reduced product

<u>Corollary 16.9.</u>

The above corollary holds also for $\{\varphi_i\}_{i\in\omega} \subseteq {}_1F^t$ as well.

<u>Proof</u>

Let $\{x_i\}_{i<n}$ be the set of free variables of φ.
$\mathfrak{A} \models \varphi$ iff $\mathfrak{A} \models \forall x_0 x_1 \dots x_{n-1}\varphi$ by the definition of ${}_1\models^t$.

This completes the proof since $\forall x_0 \dots x_{n-1}\varphi \in {}_1S^t$.

▲

Corollary 16.1o.

Let $\{\varphi_i\}_{i<\omega} \subseteq {}_1F^t$, now:

if $\bigwedge_{i\in\omega} \varphi_i \models \varphi$ then $\bigwedge_{i<n} \varphi_i \models \varphi$ for some $n \in \omega$.

Proof

$\bigwedge_{i\in\omega} \varphi_i \models \psi$ iff

$\models \psi \vee \bigvee_{i\in\omega} \neg\varphi_i$ iff /by the compactness theorem of ${}_1L^t$ - see Corollary 16.8./

$\models \psi \vee \bigvee_{i<n} \neg\varphi_i$ for some n, iff

$\bigwedge_{i<n} \varphi_i \models \psi$.

▲

Theorem 16.11.

/Generalised adequateness theorem first-order logic/

There is an algorithm ${}_1GACAL^t$ which is adequate for ${}_1L^t$ in the generalised sense, that is for any $\varphi \in {}_1F^t$ the algorithm ${}_1GACAL^t(\varphi)$ enumerates the set of all consequences of φ

With other words

$$Rg\ {}_1GACAL(\varphi) = \{\psi \in {}_1F^t : \varphi \models \psi\}$$

Proof

The proof is entirely analogous to the proof of Theorems 6.11. and 6.13. The basic idea is that for any language which has implications completeness and compactness imply generalised adequateness.

▲

The reduced products when D is an ultrafilter are useful operations since they give a strong connection between M^t and $_1L^t$. These operations are thruthvalue preserving ways of composing more models into one. Some applications are the compactness theorem and Generalised adequateness theorem. These operations have many important applications in Model Theory which are not treated here.

The following theorem states that Theorem 16.7. does not hold for $_2L^t$.

Theorem 16.12.

There is an ultrafilter D and $\langle \mathfrak{A}^i \rangle_{i\in I}, \varphi \in {}_2F^t$ such that $\mathbb{P}_D \mathfrak{A}^i \models \varphi$ and $H_\varphi \notin D$.

Proof

Let $\mathfrak{A}^0_0 \overset{d}{=} \mathfrak{A}_0^1 = \{a, b\}, \mathfrak{A}_a^0 = \mathfrak{A}_a^1 = a; \mathfrak{A}_b^0 = \mathfrak{A}_b^1 = b,$ $I \overset{d}{=} \{0,1\}$

Let D be an arbitrary ultrafilter on I.,

$P \in V_1^R, x \in V$

$\psi \overset{d}{=} (\exists x \neg P(x) \wedge P(a) \wedge P(b))$

$\varphi \overset{d}{=} \exists P \psi$

clearly $\mathbb{P}_D \mathfrak{A}^i \models \varphi$, since let $k(P) = \{\langle a,a\rangle, \langle b,b\rangle\}$.

Now $\mathbb{P}_D \mathfrak{A}^i \models \neg P(x)[k]$ if $k(x) = \langle a, b\rangle$.

In the same time $\mathfrak{A}^0 \not\models \varphi, \mathfrak{A}^1 \not\models \varphi$ and thus $H_\varphi = \emptyset$.

▲

Theorem 16.13. /Shelah isomorphism theorem/

Let $\mathfrak{A}, \mathfrak{B} \in M^t$. The following are equivalent:

(i) for all $\varphi \in {}_1F^t$ $\mathfrak{A} \models \varphi$ iff $\mathfrak{B} \models \varphi$.

(ii) There exists an ultrafilter D such that $\mathbb{P}_D\mathfrak{A}$ and $\mathbb{P}_D\mathfrak{B}$ are isomorphic.

●

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CONTENT

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